De Broglie wavelength of a macroscopic object

$$\left.\begin{aligned} m &= 5.17\,kg \\ v &= 10.2\,m/s \end{aligned}\right\} \; p = mv$$

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \cdot 10^{-34}\,Js}{5.17\,kg \cdot 10.2\,m/s}$$

$$\lambda = 1.26 \cdot 10^{-35}\,m$$

This is a very short wavelength.
The wave behavior of macroscopic objects is not very significant.

Classical, relativistic and ultra-relativistic electron

$\boxed{\lambda = \frac{h}{p}}$ for all of them, but p is calculated differently at different energies.

Classical: $KE = \frac{p^2}{2m} \Rightarrow p^2 = 2mKE$

$\Rightarrow p = \sqrt{2mKE}$ (my $KE = 59.4\,eV$)

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mKE}} = \frac{hc}{\sqrt{2mc^2 \cdot KE}} = \frac{hc}{\sqrt{2E_0 \cdot KE}}$$

$$\lambda = \frac{1240\,eVnm}{\sqrt{2 \cdot 5.11 \cdot 10^5 eV \cdot 59.4\,eV}} = 0.159\,nm$$

$$\lambda = 0.159\,nm = 1.59\,\mathring{A} = 159\,pm$$

Relativistic electron: my $KE = 1.44\,MeV$

$$E^2 = (pc)^2 + (m_0c^2)^2 = (pc)^2 + E_0^2 \Rightarrow$$

$$\Rightarrow E^2 - E_0^2 = (pc)^2 \Rightarrow \sqrt{E^2 - E_0^2} = pc$$

$$E = E_0 + KE \Rightarrow E^2 = E_0^2 + 2E_0 \cdot KE + KE^2$$

$$E^2 - E_0^2 = 2E_0 \cdot KE + KE^2$$

Relativistic electron continued:

$$pc = \sqrt{2E_0 \cdot KE + KE^2}$$

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\sqrt{2E_0 \cdot KE + KE^2}}$$

$$\lambda = \frac{1240\,\text{eVnm}}{\sqrt{(2 \cdot 0.511 \cdot 1.44 + 1.44^2) \cdot (10^6\,\text{eV})^2}}$$

$$\lambda = \frac{1240\,\text{eVnm}}{1.88 \cdot 10^6\,\text{eV}} = 658 \cdot 10^{-6}\,\text{nm}$$

$$\lambda = 0.658\,\text{pm}$$

Ultrarelativistic electron:

$KE \gg E_0$ my $KE = 2.24\,\text{GeV}$

$$\lambda = \frac{hc}{\sqrt{KE(2E_0 + KE)}} \approx \frac{hc}{\sqrt{KE^2}} = \frac{hc}{KE}$$

$$\lambda = \frac{1240\,\text{eVnm}}{2.24 \cdot 10^9\,\text{eV}} = 553 \cdot 10^{-9}\,\text{nm}$$

$$\lambda = 0.553\,\text{fm}$$

Protons from the Tevatron

$E_0 = 938\ \text{MeV}$: rest energy of the proton

$KE = 554\ \text{GeV}$: kineti energy of these protons. This is a very high energy, we need to use relativistic mechanics:

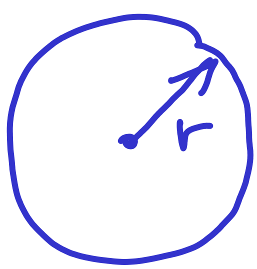
$$pc = \sqrt{KE^2 + 2KE \cdot E_0}$$

Since the kinetic energy is so huge ($554\ \text{GeV} \gg 0.938\ \text{GeV}$), we can use the $pc \approx KE$ approximation:

$$\lambda = \frac{h}{p} = \frac{hc}{pc} \cong \frac{hc}{KE} = \frac{1240\ \text{eV nm}}{554\ \text{GeV}}$$

$$\lambda = 2.24 \cdot 10^{-9}\ \text{nm} = 2.24 \cdot 10^{-3}\ \text{fm} = 2.24\ \text{am}$$

(a = atto)



proton

$$r = 0.94\ \text{fm} = 940\ \text{am}$$

$$\frac{r}{\lambda} = \frac{940\ \text{am}}{2.24\ \text{am}} \cong 420$$

In theory we can probe 1/420 of the proton radius with protons accelerated to this energy.

$1/420 = 2.38 \cdot 10^{-3}$ or 0.238% of the r.

Energy uncertainty of a particle

Lifetime: $\tau = 47.9\,as = 47.9 \cdot 10^{-18}\,s$

Rest mass: $m_0 = 124.6\ MeV/c^2 \Rightarrow$

$\Rightarrow$ rest energy: $E_0 = 124.6\ MeV$.

Time uncertainty of this particle:

$\Delta t \cong \tau$

Heisenberg Uncertainty Principle:

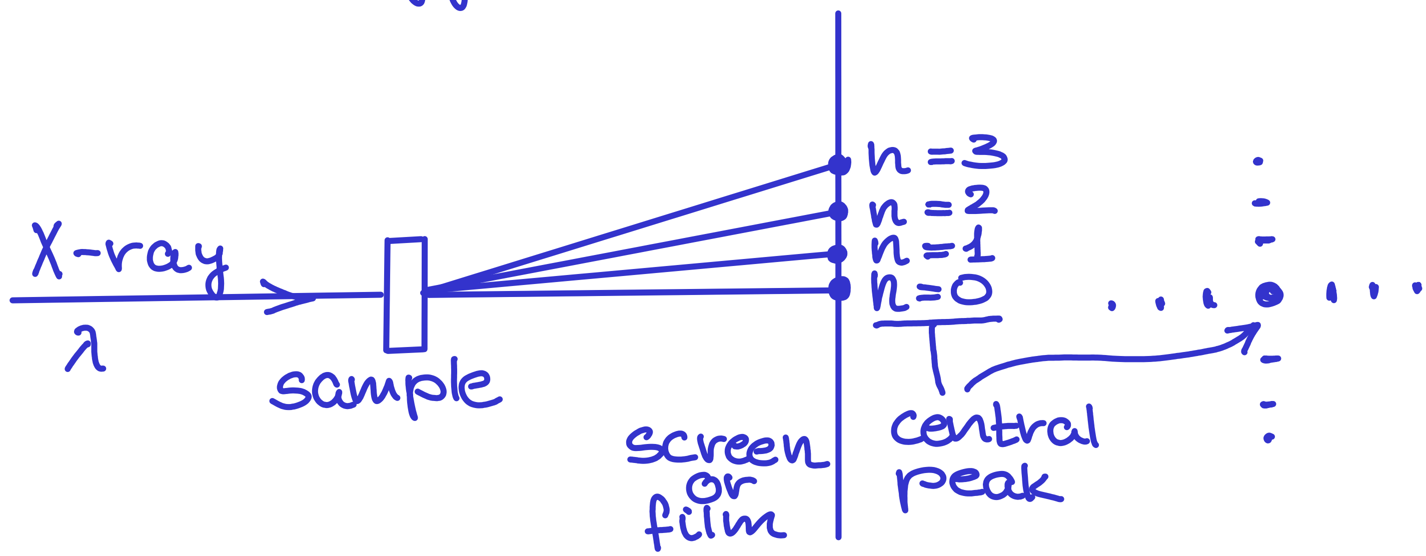
$$\Delta t \cdot \Delta E \geq \frac{\hbar}{2}$$

$$\Delta E \geq \frac{\hbar}{2\Delta t} \cong \frac{\hbar}{2\tau}$$

$\hbar = 6.5821 \cdot 10^{-16}\ eVs$

$$\Delta E \geq 6.87\ eV$$

Bragg's law



Bragg's law:

$$2d\sin\theta = n\lambda \qquad n=1,2,3,4,\dots$$

$$\left.\begin{aligned} 2d\sin\theta_1 &= 1\cdot\lambda \\ 2d\sin\theta_2 &= 2\cdot\lambda \\ 2d\sin\theta_3 &= 3\cdot\lambda \end{aligned}\right\} \quad \frac{\sin\theta_n}{n} = \frac{\lambda}{2d}$$

$$\frac{\sin\theta_1}{1} = \frac{\sin\theta_2}{2} = \frac{\sin\theta_3}{3} = \dots$$

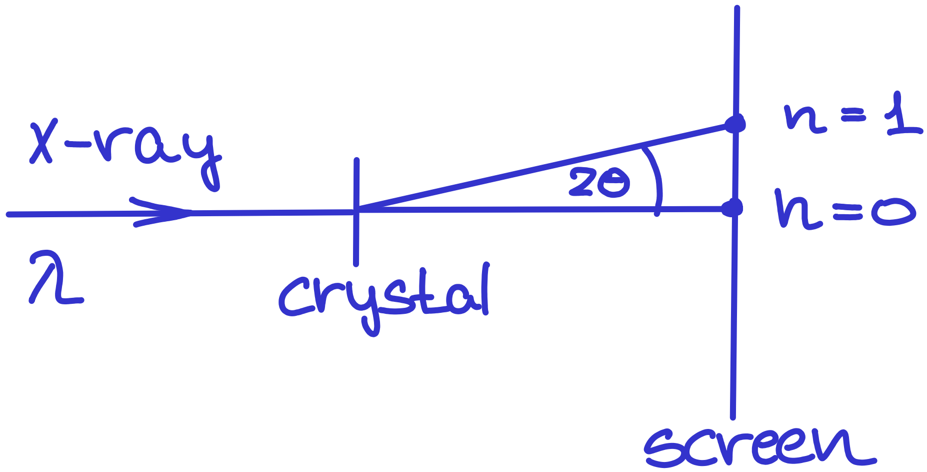
$$\sin\theta_2 = 2\sin\theta_1 \Rightarrow \theta_2 = \arcsin(2\sin\theta_1)$$

$$\sin\theta_3 = 3\sin\theta_1 \Rightarrow \theta_3 = \arcsin(3\sin\theta_1)$$

If $\theta_1 = 17.3^\circ$, then $\theta_2 = 36.5^\circ$

and $\theta_3 = 63.1^\circ$

Lattice constant of a crystal



X-ray wavelength: $\lambda = 0.890\,nm$
and $\theta = 17.8°$.

Bragg's law:

$$2d \cdot \sin\theta = n\lambda \qquad n = 1, 2, 3, \ldots$$

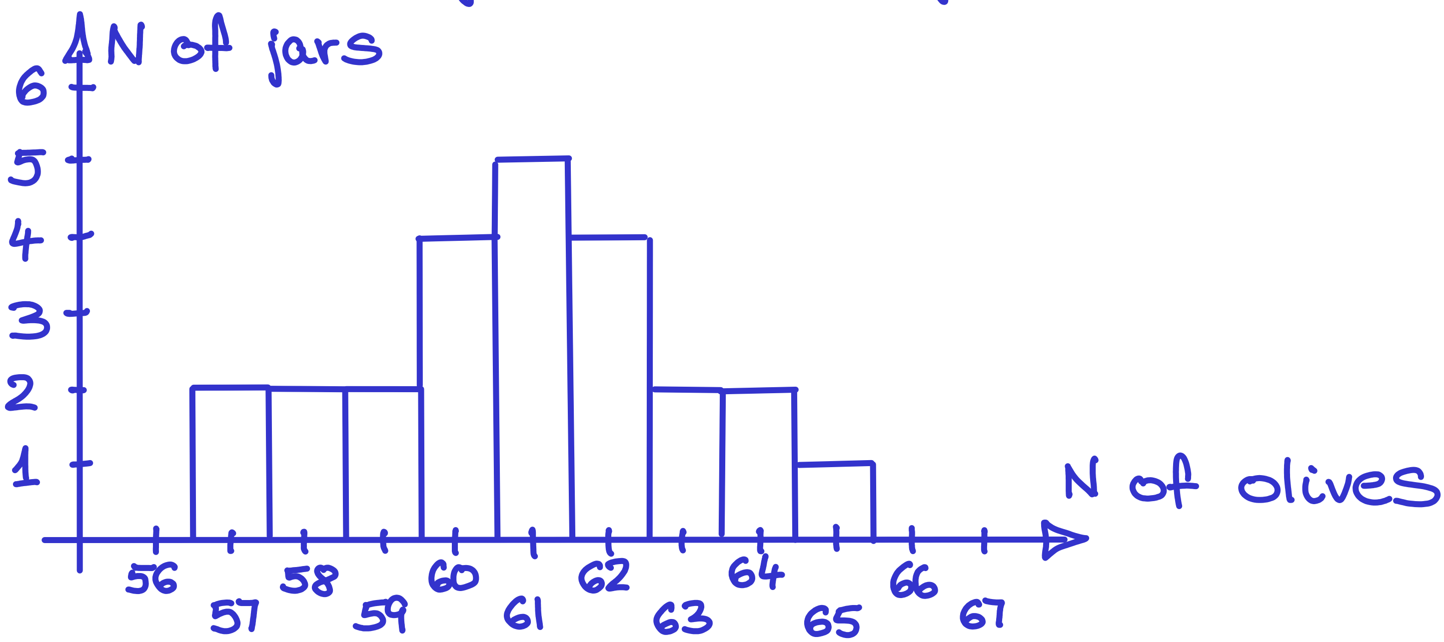
$$d = \frac{n\lambda}{2 \cdot \sin\theta}$$

$$d = \frac{1 \cdot 0.890\,nm}{2 \cdot \sin 17.8°} = 1.46\,nm$$

For simple cubic lattice (sc):

$d = d_1 = D$ where D is the lattice constant. This $D = 1.46\,nm$ is larger than the lattice constant of simple metals (Ni: $0.215\,nm = 2.15\,Å$). This is probably a molecular crystal.

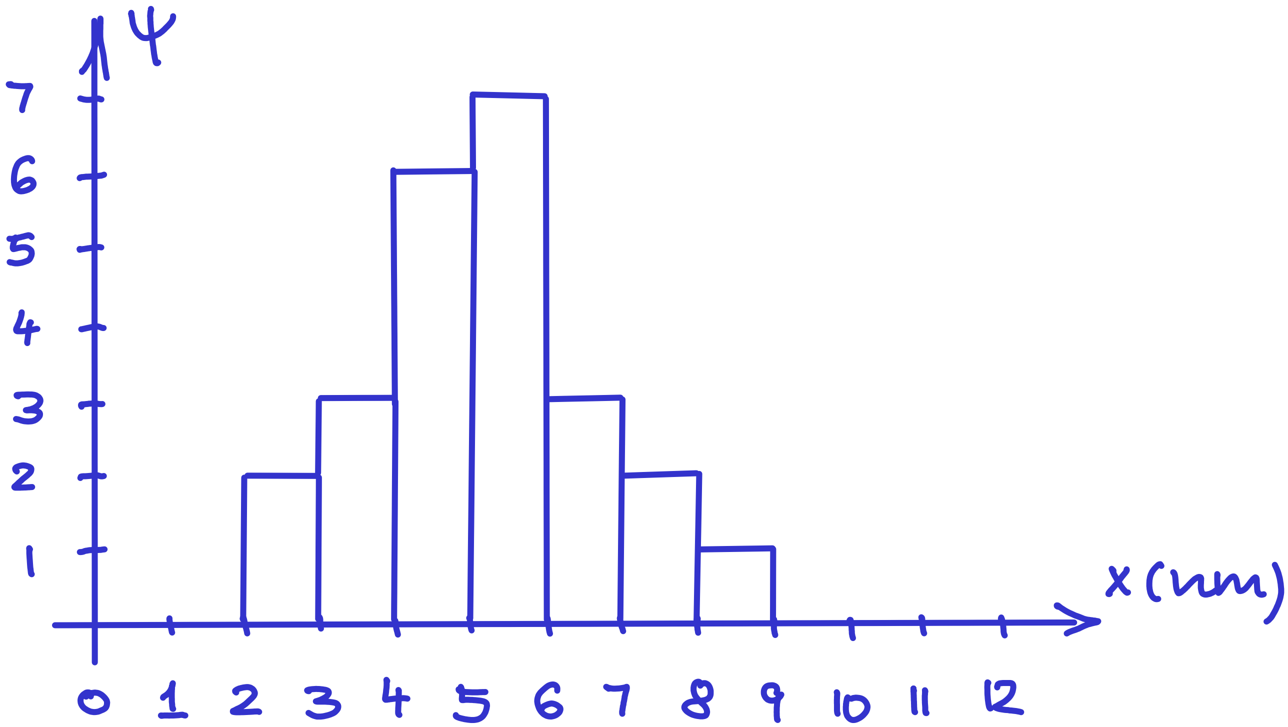
Quality control engineer



Sample size: $2+2+2+4+5+4+2+2+1=24$

$$P(N_{olives} < 59) = \frac{2+2}{24} = \frac{1}{6} = 0.167 \ (=16.7\%)$$

Wave function in one dim. (a)



ψ: 2, 3, 6, 7, 3, 2, 1

ρ: 4, 9, 36, 49, 9, 4, 1 $\quad (\rho = \psi^* \psi = |\psi|^2)$

Total : $\Sigma = 112$

$$P(4 \leq x \leq 8) = (36 + 49 + 9 + 4) / 112 =$$
$$= 98 / 112 = 0.875$$

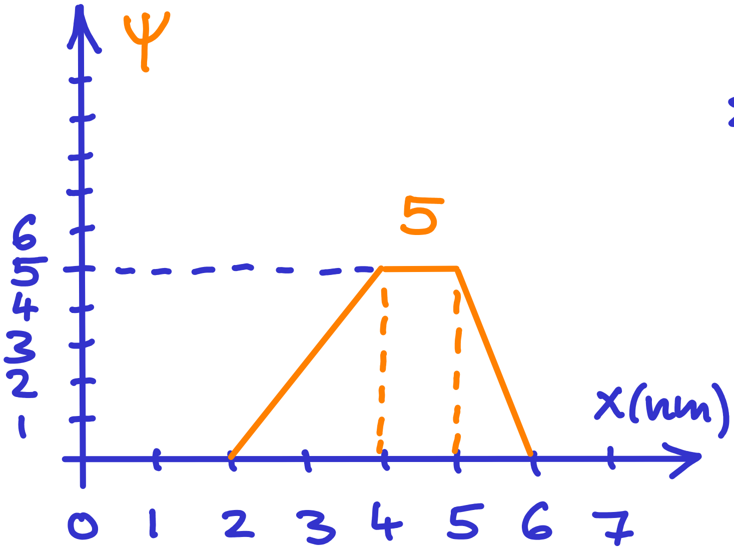
(Normalization constant : $A = \frac{1}{\sqrt{112}}$)

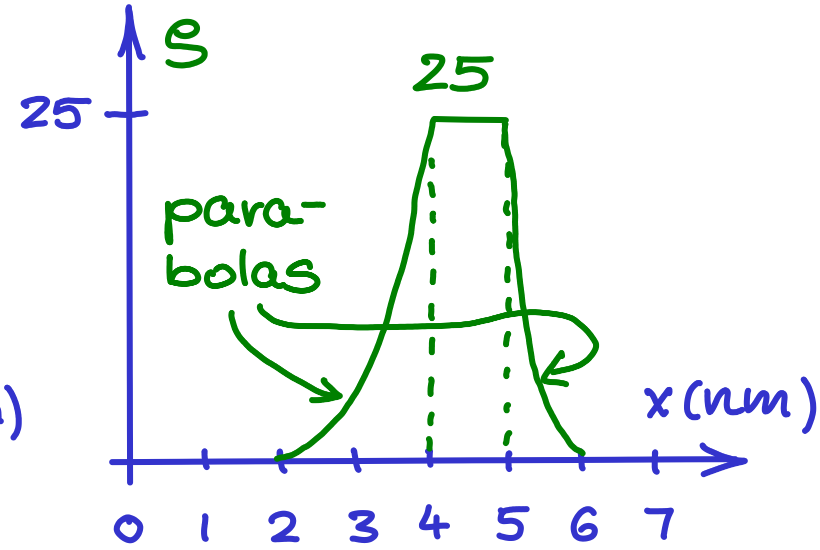
Unit of the wavefunction in 1D:

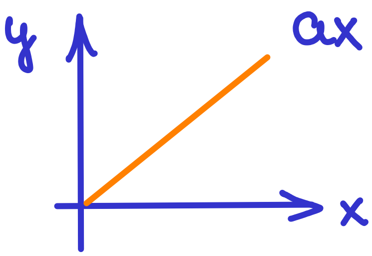
$$1 = \int \rho(x)\, dx = \int |\psi|^2\, dx$$

$$[\rho] = \frac{1}{m} \qquad [\psi] = \frac{1}{\sqrt{m}}$$

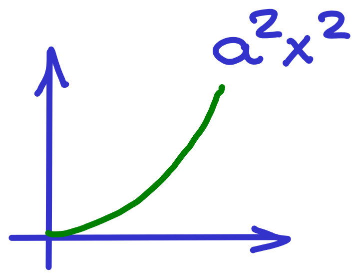
Wave function in one dim. (b)







$\Rightarrow$



$$\int a^2x^2\,dx = a^2\cdot\frac{x^3}{3}$$

Normalization: $1 = \int \rho(x)\,dx$

$$\left.\begin{aligned}
\Sigma_1 &= \left(\frac{5}{2}\right)^2\cdot\frac{2^3}{3} = \frac{50}{3} \cong 16.7\\
\Sigma_2 &= 25\cdot 1 = 25\\
\Sigma_3 &= 5^2\cdot\frac{1^3}{3} = \frac{25}{3} \cong 8.33
\end{aligned}\right\}\ \text{Total area: } \Sigma = 50$$

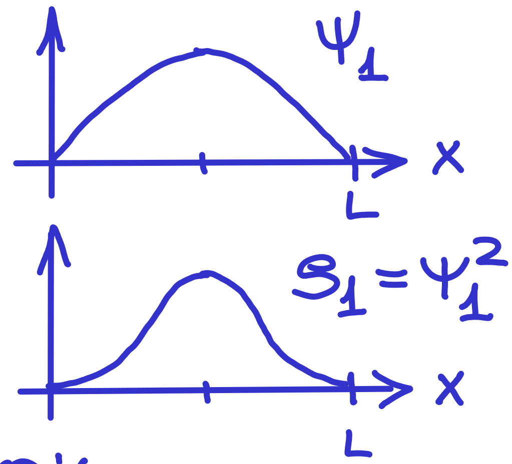
Normalization constant: $A = \frac{1}{\sqrt{50}}$

$$P(2 \le x \le 4) = \frac{\Sigma_1}{\Sigma} = \frac{50/3}{50} = \frac{1}{3} = 0.333$$

Probabilities in an infinite square well

$$\psi_1 = \sqrt{\frac{2}{\pi}} \cdot \sin\left(\frac{\pi x}{L}\right)$$

$$\left(\psi_n = \sqrt{\frac{2}{\pi}} \cdot \sin\left(\frac{n\pi x}{L}\right)\right)$$



P(left half) = 0.5 = 50%
P(right half) = 0.5 = 50%

P(1st third) $\neq 1/3$
P(2nd third) $\neq 1/3$
P(3rd third) $\neq 1/3$

Trig: $\sin^2(\alpha) = \dfrac{1-\cos(2\alpha)}{2}$

$$P(\text{1st third}) = \int_0^{L/3} \rho_1\, dx = \frac{2}{\pi}\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx$$

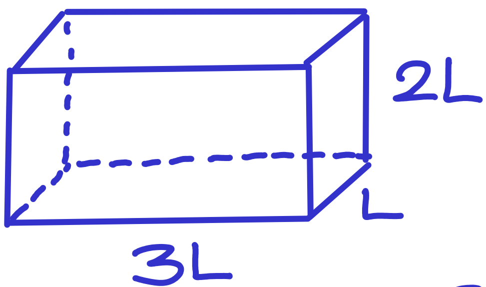
$$= \frac{2}{\pi}\int_0^{\pi/3} \sin^2(x)\, dx = \frac{2}{\pi}\int_0^{\pi/3} \frac{1-\cos(2x)}{2}\, dx =$$

$$= \frac{2}{\pi}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]\Bigg|_0^{\pi/3} =$$

$$= \frac{2}{\pi}\left(\frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right) = \underbrace{\frac{1}{3}}_{33.33\%} - \underbrace{\frac{\sqrt{3}}{4\pi}}_{13.78\%} = 0.1955$$

P(3rd third) = P(1st third)
P(2nd third) = 1 − 2 · P(1st) = 1 − 2 · P(3rd)

Box with sides $L, 2L, 3L$



$$E_{n_1,n_2,n_3} = \frac{h^2}{8m}\left(\frac{n_1^2}{L^2} + \frac{n_2^2}{(2L)^2} + \frac{n_3^2}{(3L)^2}\right)$$

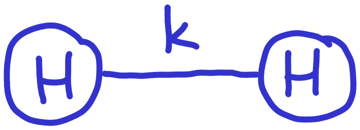
$$= \frac{h^2}{8mL^2}\left(n_1^2 + \frac{n_2^2}{4} + \frac{n_3^2}{9}\right)$$

n_1	n_2	n_3			
1	1	1	gs	$1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$	#1
1	1	2		$1 + \frac{1}{4} + \frac{4}{9} = \frac{61}{36}$	#2
1	2	1		$1 + \frac{4}{4} + \frac{1}{9} = \frac{76}{36}$	#3
2	1	1		$4 + \frac{1}{4} + \frac{1}{9} = \frac{157}{36}$	
1	1	3		$1 + \frac{1}{4} + \frac{9}{9} = \frac{81}{36}$	#4
1	3	1		$1 + \frac{9}{4} + \frac{1}{9} = \frac{121}{36}$	
3	1	1		$9 + \frac{1}{4} + \frac{1}{9} = \frac{337}{36}$	

Box with sides L, 2L, 3L (cont'd)

1	2	2	$1+\frac{4}{4}+\frac{4}{9}=\frac{88}{36}$ #5
2	1	2	$4+\frac{1}{4}+\frac{4}{9}=\frac{169}{36}$
2	2	1	$4+\frac{4}{4}+\frac{1}{9}=\frac{184}{36}$
2	2	2	$4+\frac{4}{4}+\frac{4}{9}=\frac{196}{36}$

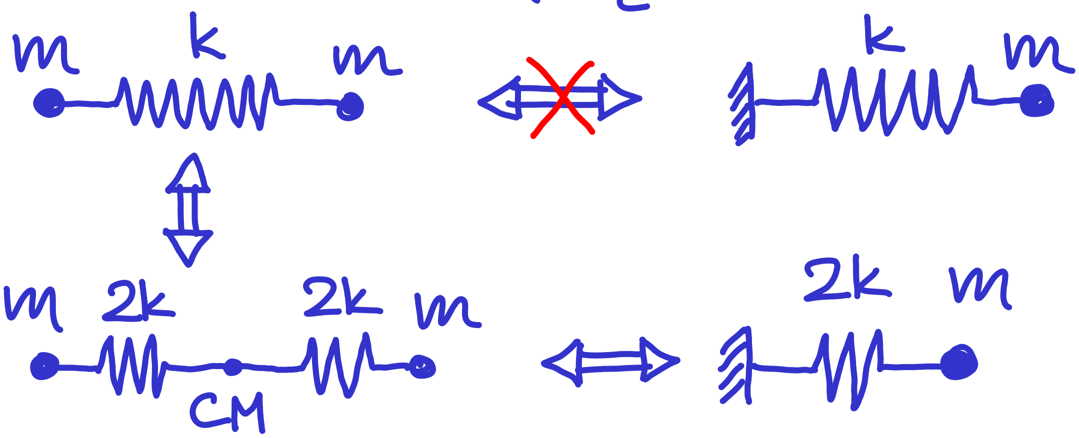
Hydrogen molecule



$k = 1100\ N/m$

$m_p = 1.6726 \cdot 10^{-27}\ kg$: proton

$(m_e = 0.0009 \cdot 10^{-27}\ kg$: electron$)$



$$\omega = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\frac{2k}{m}}$$

Reduced mass:

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \Rightarrow \mu = \frac{m}{2}$$

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k}{\frac{m}{2}}} = \sqrt{\frac{2k}{m}}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \ ; \quad E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{2k}{m}}$$

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{2k}{m}}$$

$$\lambda = \frac{c}{f} = \frac{c}{\frac{\Delta E}{h}} = \frac{hc}{\Delta E} = \frac{hc}{\hbar\sqrt{\frac{2k}{m}}} = 2\pi c\sqrt{\frac{m}{2k}}$$

$$\left(T = \frac{\lambda}{c} = 2\pi\sqrt{\frac{m}{2k}} = 2\pi\sqrt{\frac{\mu}{k}}\right)$$

Electron energies in 1D and 3D wells

Electron: $m_e = 511\ \text{keV}/c^2$

Width of the well: $L = 7.0\ \text{Å} = 0.7\ \text{nm}$

$\boxed{1D}$: $$E_n = n^2 \frac{h^2}{8mL^2} = n^2 \frac{(hc)^2}{8m_ec^2L^2} = n^2 E_1$$

$$E_1 = \frac{(1240\,\text{eV nm})^2}{8 \cdot 511{,}000 \cdot (0.7\,\text{nm})^2} = 0.767\ \text{eV}$$

$$E_{gs} = E_1 = 0.767\ \text{eV}$$
$$E_2 = 2^2 \cdot E_1 = 3.07\ \text{eV}$$
$$E_3 = 3^2 \cdot E_1 = 6.91\ \text{eV}$$

$\boxed{3D}$: $$E_{n_1,n_2,n_3} = \frac{h^2}{8m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

Cubic box: $$E_{n_1,n_2,n_3} = \frac{h^2}{8mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$$

$$E_{gs,3D} = 3 \cdot E_{gs,1D} = 3 \cdot 0.767\,\text{eV} = 2.30\,\text{eV}$$
$$E_{1st,3D} = 2 \cdot E_{gs,3D} = 4.60\ \text{eV}$$

gs: $1\,1\,1 : 3$

$$\left.\begin{array}{c} 2\,1\,1 \\ 1\,2\,1 \\ 1\,1\,2 \end{array}\right\} 6 \qquad \left.\begin{array}{c} 2\,2\,1 \\ 2\,1\,2 \\ 1\,2\,2 \end{array}\right\} 9 \qquad \left.\begin{array}{c} 3\,1\,1 \\ 1\,3\,1 \\ 3\,1\,1 \end{array}\right\} 11$$

$2\,2\,2\ \}\ 12$

$$\left.\begin{array}{c} 1\,2\,3 \\ 1\,3\,2 \\ 2\,1\,3 \\ 2\,3\,1 \\ 3\,1\,2 \\ 3\,2\,1 \end{array}\right\} 14$$