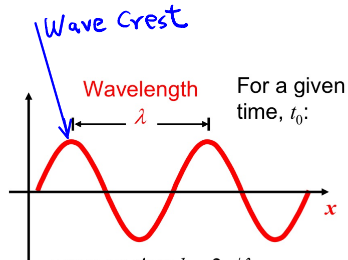
Relativistic Doppler Effects

Relativistic Doppler Effect

(i) Wave

Definitions

Spatial quantities:



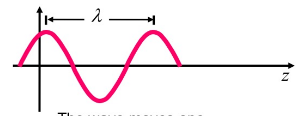
For a given time, t_0:

wave vector: $k = 2\pi/\lambda$
wave number: $\kappa = 1/\lambda = k/2\pi$

The Velocity

How fast is the wave traveling?

Velocity is a reference distance divided by a reference time.



The wave moves one wavelength, λ, in one period, T

The velocity is the wavelength / period:

$$\boxed{c = \lambda / T = \lambda f}$$

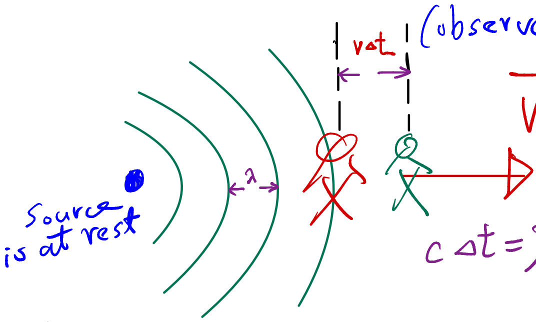
In terms of the k-vector, $k = 2\pi/\lambda$, and the angular frequency, $\omega = 2\pi f$, this is:

$$\boxed{c = \omega / k}$$

Units: Wavelength $[\lambda] = m$ (meters)
period $[T] = \text{sec}$
velocity $[c] = m/\text{sec}$
frequency $[f] = \frac{1}{\text{sec}}$ $\quad \left(f = \frac{1}{T}\right)$

- Source: Sends signal (emitter)
- Observer: Receives signal
- Relative motion between them

(observer is moving away)



(measured at the source, at the stationary frame)

$$c\Delta t = \lambda + v\Delta t \Rightarrow \Delta t = \frac{\lambda}{c-v}$$

According to the observer, due to time dilation,

$$\Delta t' = \frac{\Delta t}{\gamma} \quad \left(\text{for } \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \geq 1\right)$$

$$= \sqrt{1-\frac{v^2}{c^2}}\left(\frac{\lambda}{c-v}\right) = \frac{\lambda}{c}\left(\frac{\sqrt{c^2-v^2}}{c-v}\right)$$

$$= \frac{\lambda}{c}\left(\frac{\sqrt{c+v}\,\cancel{\sqrt{c-v}}}{\sqrt{c-v}\,\cancel{\sqrt{c-v}}}\right) = \frac{\lambda}{c}\left(\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}\right)$$

Since $f = \frac{1}{T}$ $\Rightarrow$ $f_{obs} = \frac{1}{\Delta t'}$, $f_{emit} = \frac{1}{\Delta t}$

$$\Rightarrow f_{obs} = \left(\frac{c}{\lambda}\right)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = f_{emit}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

$$\Rightarrow f_{obs} = f_{emit}\sqrt{\frac{1-\beta}{1+\beta}}, \quad \text{for } \beta \equiv \frac{v}{c} \leq 1$$

If the observer is approaching to the source, then change v to $(-v)$ in the above equation

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For $v \ll c$, than $\boxed{\beta \ll 1,}$

$$\sqrt{1-\beta} = (1-\beta)^{1/2} \simeq 1 - \frac{1}{2}\beta + \cdots$$

$$\sqrt{1+\beta} = (1+\beta)^{1/2} \simeq 1 + \frac{1}{2}\beta + \cdots$$

$$\Rightarrow \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \simeq \frac{1-\frac{1}{2}\beta}{1+\frac{1}{2}\beta} \overset{\beta \ll 1}{\simeq} \left(1-\frac{1}{2}\beta\right)\left(1-\frac{1}{2}\beta\right) \simeq 1-\beta$$

$$\Rightarrow f_{obs} \simeq f_{emit}(1-\beta), \quad \text{for } \beta = \frac{v}{c} \ll 1$$

this agrees with classical Doppler effect.

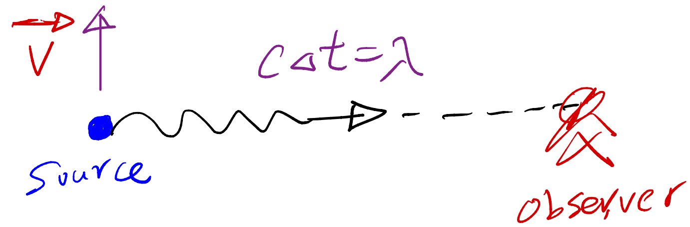
Note: For classical Doppler effect:

$\Delta t' = \Delta t$ (classical kinematics) $\quad \beta \equiv \frac{v}{c}$

$$\Rightarrow \Delta t' = \frac{\lambda}{c-v} = \frac{\lambda}{c}\left(\frac{1}{1-\beta}\right),$$

$$\Rightarrow f_{obs} = f_{emit}(1-\beta).$$

If the source emits a plane wave along the x-direction, while moving along the y-direction with the speed v, then



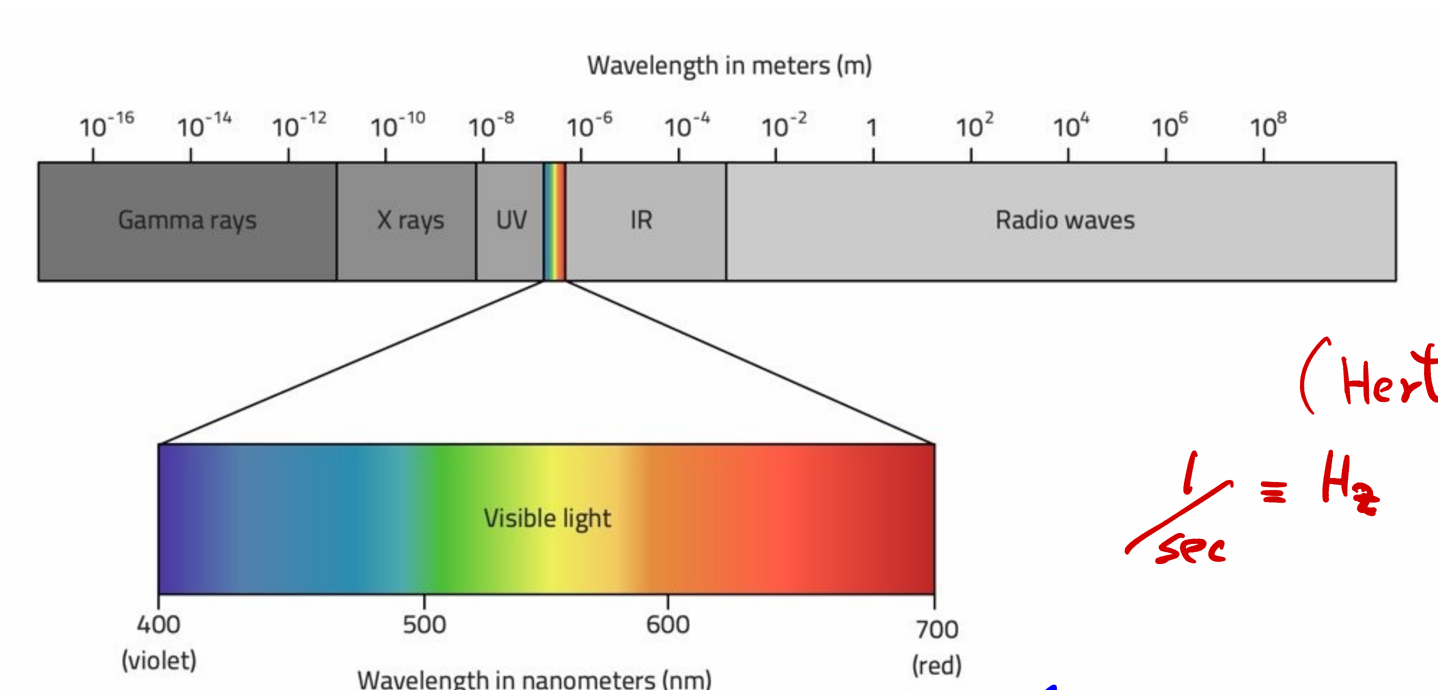
because the source's clock is time dilated, the light is redshifted, with smaller frequency.

$$f_{obs} = \frac{f_{emit}}{\gamma} = f_{emit}\sqrt{1-\beta^2}, \quad \left(\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \geq 1, \quad \beta \equiv \frac{V}{c} < 1 \right)$$

Electromagnetic Spectrum

$$c = \lambda f$$

$$c = 3 \times 10^8 \, m/sec$$

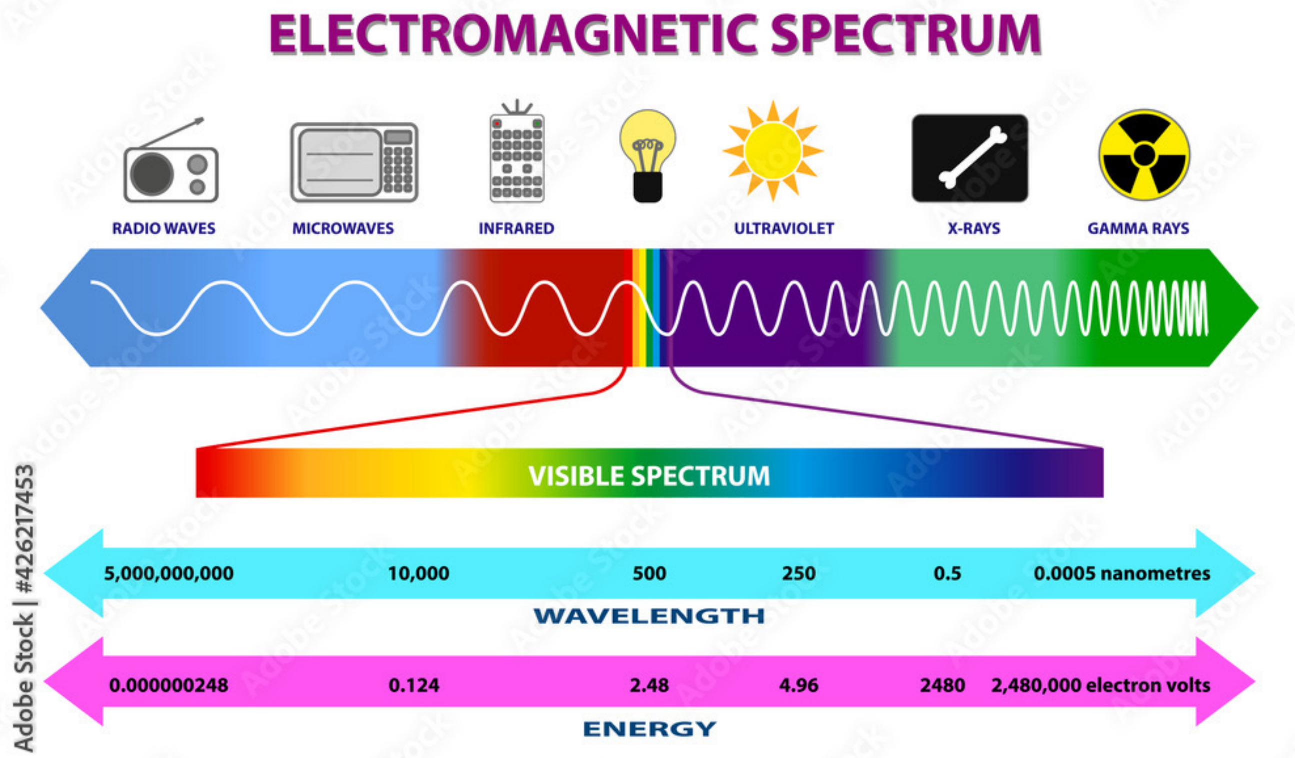


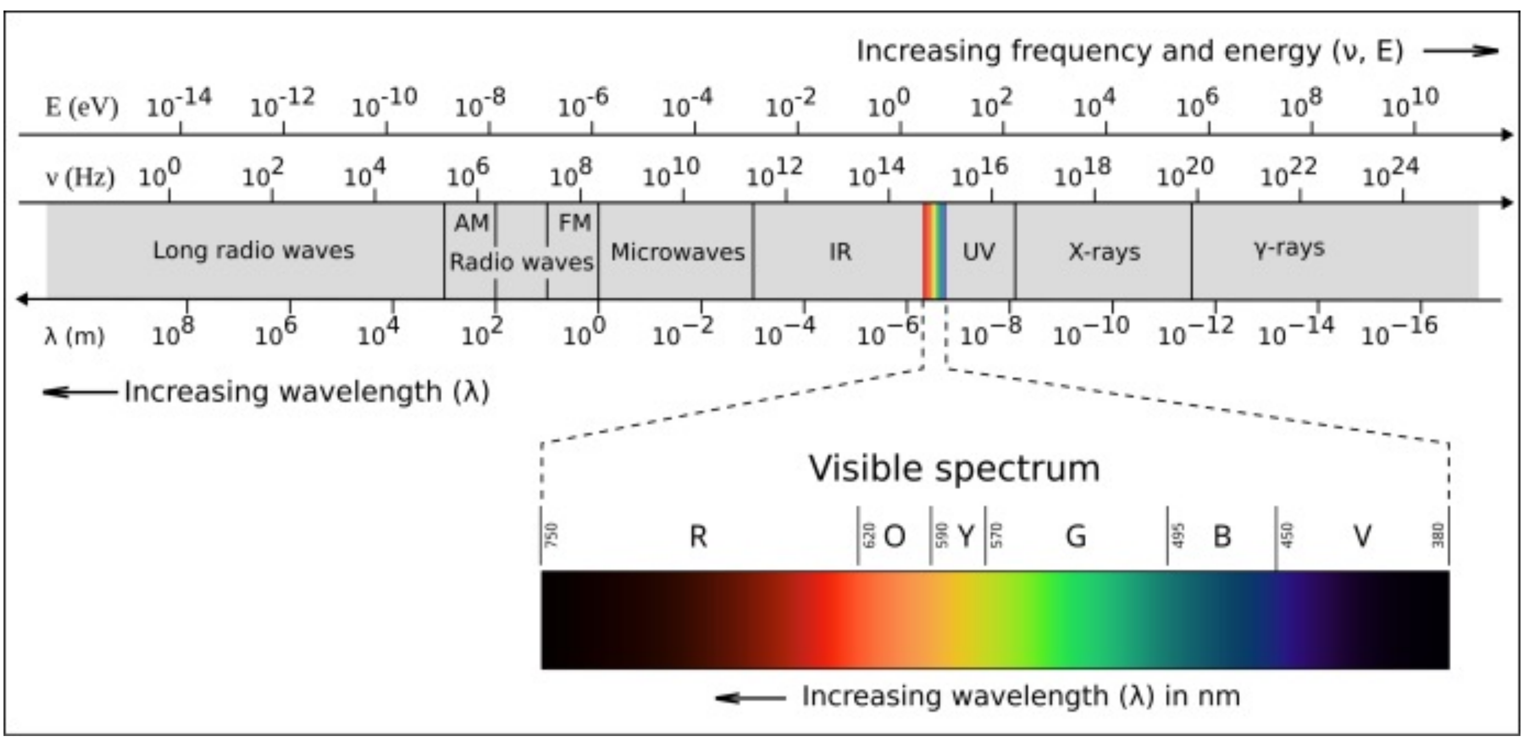
(Hertz)

$$\frac{1}{sec} \equiv Hz$$

$1 \, nm = 10^{-9} m$, $\quad f = \frac{c}{\lambda} \Rightarrow$ For $\lambda = 600 \, nm$,

$$f = \frac{3 \times 10^8 \, m/sec}{600 \times 10^{-9} \, m} = \frac{1}{2} \times 10^{15} /sec = 0.5 \times 10^{15} \, Hz$$





$\lambda = \frac{c}{f} \Rightarrow$ f increased $\Rightarrow \lambda$ decreased blueshift

f decreased $\Rightarrow \lambda$ increased redshift

Twin Paradox

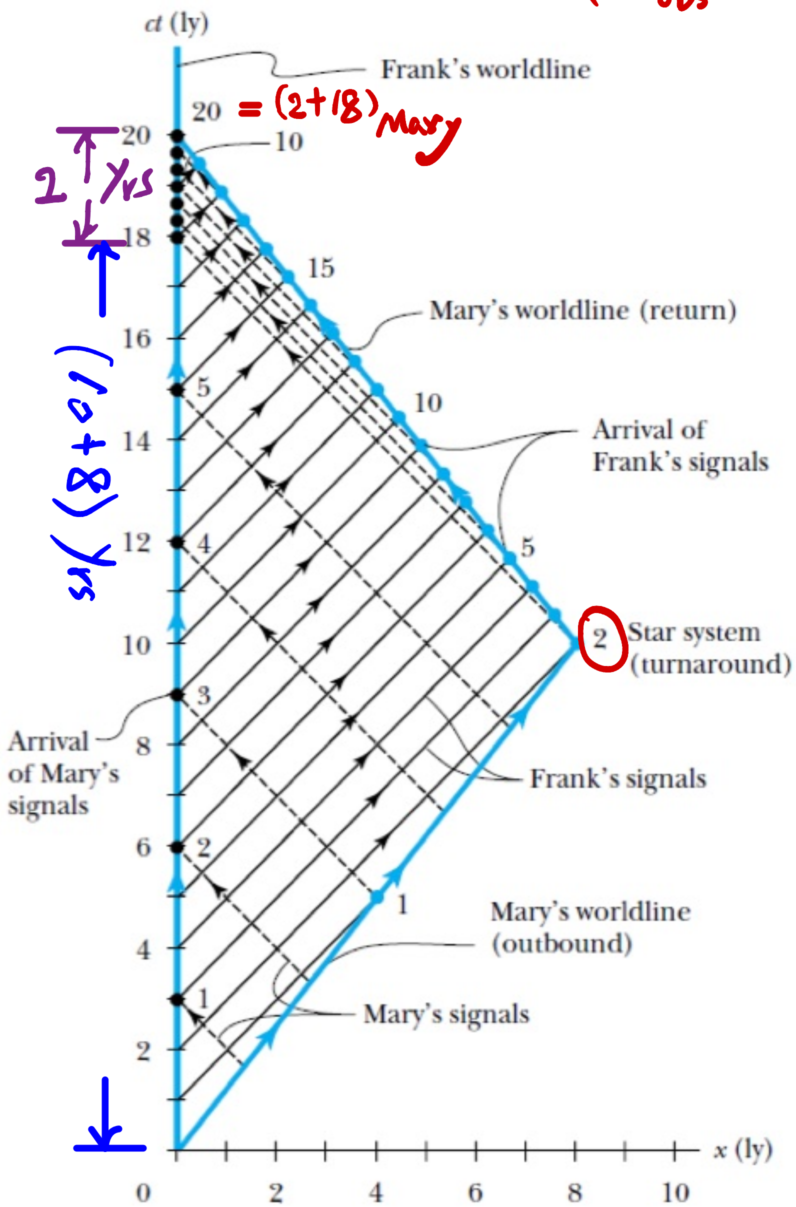
Relativistic Doppler effect

$$\left(f_{obs} = f_{emit}\sqrt{\frac{1\mp\beta}{1\pm\beta}} \right)$$

$V = 0.8c = \frac{4}{5}c$, $\beta = \frac{v}{c} = \frac{4}{5}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{5}{3}$

outbound: $\sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$ (1 flash per 3 years)

Inbound: $\sqrt{\frac{1+\beta}{1-\beta}} = 3$ (3 flashes per year)



According to Frank's viewpoints

(It takes 10 years to reach the turning point, and 8 years for the flash emmited at that point to reach the Earth)

	Mary's experience	Frank's experience
Outbound:	$(6\text{ yrs})\left(\frac{1}{3}\right) = 2$	$(10+8)\text{ yrs}\cdot\left(\frac{1}{3}\right) = \frac{18}{3} = 6$
Inbound:	$(6\text{ yrs})(3) = 18$	$(20-10-8)\text{ yrs}\cdot(3) = 6$
	⇒ Mary sees Frank $2+18 = 20$ years older	⇒ Frank sees Mary $6+6 = 12$ years older

Hence, Mary is younger than Frank when they meet again after the journey.