Special Relativity

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PHY 215, S525

Galileo: Inertial System

— If an object is moving with a constant velocity $\vec{v}$ (which contains magnitude (speed) and direction). $\rightarrow v$

vector " $\rightarrow$ "

then it will remain in that velocity, when no external force.

"At rest" is for $\vec{v} = 0$.

Newton: Three laws:

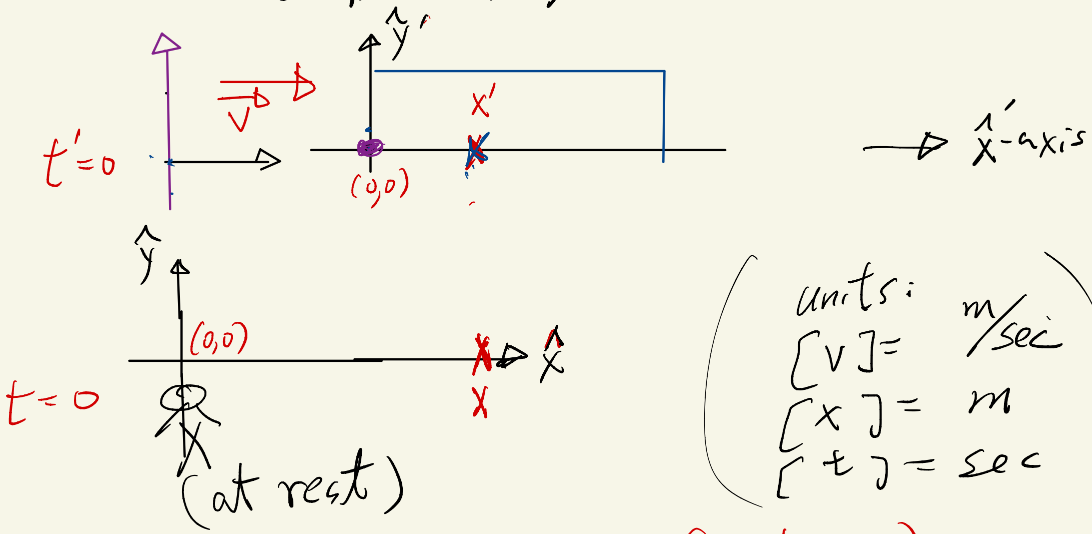
① Inertial system, when no force.
The law of physics (for describing object motion) is the same in all inertial systems.

② With force, $\vec{F} = m\vec{a} = \dfrac{d(m\vec{v})}{dt}$

$$\left(\vec{v} = \frac{d\vec{x}}{dt} \right)$$

③ reaction = (action, but with opposite direction)

Consider two inertial systems.



units:
$[v] = m/sec$
$[x] = m$
$[t] = sec$

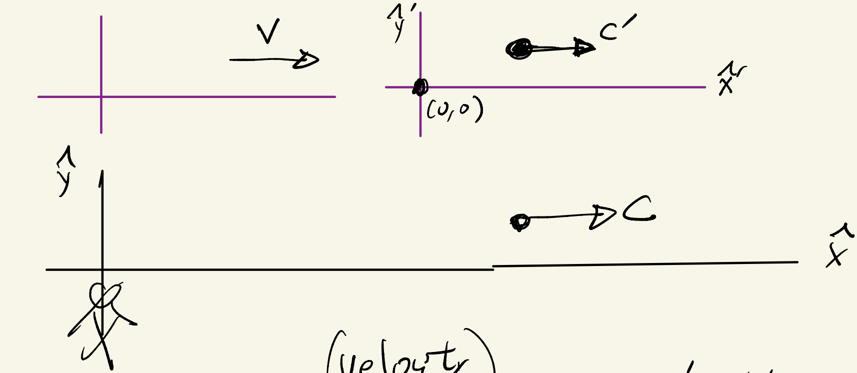
Newton: Time is absolute. $(t'=t)$

$x' = x - vt$	$x = x' + vt$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = t$	$t = t'$

Galilean transformation

To describe an event, we need to specify both position and time, (and ...).

(x, y, z, t)



$c' = c - V$ (velocity) $c = c' + V$

What if c and c' are referring to the speed of Light ?

$\Rightarrow$ $c \neq c'$ for $V \neq 0$

Maxwell combined electricity & Magnetism.

$\Rightarrow$ Newton's law for describing classical mechanics (kinematics)

Maxwell's equations for describing E&M interactions

$\Rightarrow$ showed that all the electromagnetic waves move in vacuum with the same speed. $\leftarrow 3\times10^8$ m/sec

(Light is one of the EM wave.)

$\Rightarrow$ Light moves with the same speed (in vacuum) in all inertial systems.

Einstein: The principle of relativity:

Ⓐ All the physics laws (including classical mechanics and Electromagnetic interactions)

should be the same in All the inertial systems.

⇒ There is no single absolute reference frame.

Ⓑ Speed of light (in vacuum)

is the same in all inertial system.

Consequences:

① $t' \neq t$, Time is Not absolute

② Time dilation

③ (Lorentz) length Contraction.

⇒ No speed of any object can exceed the speed of light

How to see $t' \neq t$ (for $\vec{v} \neq 0$)?

① $x' = \gamma(x - vt)$ with ($\gamma \to 1$, as $v \to 0$)

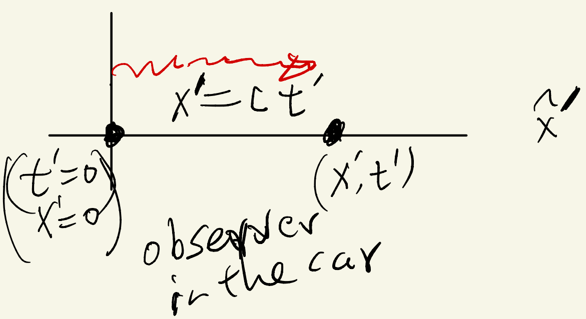
$x = \gamma'(x' + vt')$ with ($\gamma' \to 1$, as $v \to 0$)

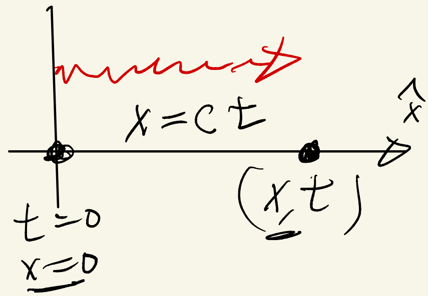
Based on Ⓐ, we require $\boxed{\gamma' = \gamma}$. independent of inertial system. (reference frame)

② Based on Ⓑ, we have

$x' = ct'$ and $x = ct$

the same





Math is the natural language to describe Nature!

$$x' = \gamma(x - vt)$$
$$\cancel{ct'} = \gamma(ct - vt)$$
$$= \gamma(c-v)t$$
$$= ct\,\gamma\left(1 - \frac{v}{c}\right)$$

$$x = \gamma(x' + vt')$$
$$ct = \gamma(ct' + vt')$$
$$= \gamma(c+v)t'$$
$$= ct'\gamma\left(1 + \frac{v}{c}\right)$$

$$= \cancel{ct'}\,\gamma\left(1+\frac{v}{c}\right)\gamma\left(1-\frac{v}{c}\right)$$
$$\Rightarrow 1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow \gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$$
$$\Rightarrow \boxed{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad \left(\gamma = 1 \text{ when } v = 0\right)$$

Note: v cannot be larger than c, because of $\sqrt{1 - v^2/c^2}$.

$x' = \gamma(x - vt)$, using $x' = ct'$, $x = ct$ and $t = \frac{x}{c}$

$$\cancel{c}t' = \gamma\left(ct - \frac{vx}{c}\right)$$
$$= \cancel{c}\gamma\left(t - \frac{vx}{c^2}\right)$$
$$\Rightarrow \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)}, \quad \left(\text{with } \gamma = 1 \text{ and } t' = t, \text{ when } v = 0\right)$$

Relativistic Kinematics is relevant when v is close to the speed of light c.

Transformation of coordinates

Galileo-Newton

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Lorentz-Fitzgerald

$$x' = \gamma(x - \beta c t)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \beta \frac{x}{c}\right)$$

$\beta = \frac{v}{c}$: speed

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$: gamma factor

$\vec{F} = m \cdot \vec{a}$ is invariant under these.

The Maxwell equations are invariant under these, same is for $\vec{F} = \frac{d(m\vec{v})}{dt}$

For $v \ll c$ ($\beta \approx 0$, $\gamma \approx 1$)

classical Kinematics

For all v ($v < c$)

Relativistic Kinematics

Time dilation

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

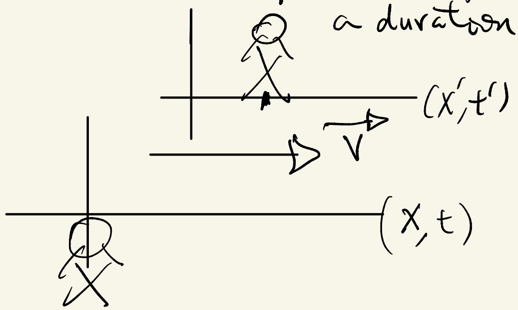
$$\Rightarrow \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \boxed{\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)}$$

light a candle for a duration ($\Delta t'$), with ($\Delta x' = 0$)



$$\Rightarrow \boxed{(\Delta t) = \gamma(\Delta t')}$$

with $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1$ for $v \neq 0$

when v is close to c, say, $v = 0.99c$,

then $\gamma = \frac{1}{\sqrt{1 - (0.99)^2}} \gg 1$

$\Rightarrow$ Time dilation.

Cosmic ray (muons)

① muon is an elementary particle, similar to electon, but with mass 200 times of electron's.

$m_{\mu} = 200\, m_e = 0.1\ \mathrm{GeV}/c^2$, ($\mathrm{GeV} = 10^9$ electron-Volt)

$c =$ Speed of light $= 3 \times 10^8\ m/sec$

② The energy of cosmic muon detected on earth is about 4 GeV.

③ the lifetime of muon has been measured to be $2.2\,\mu s = 2.2 \times 10^{-6}$ sec.

④ The height of the atmosphere of earth is about $15\ km = 15 \times 10^3\ m$

⑤ The speed of muon can be calculated by using

$\boxed{E = mc^2}$, with $\boxed{m = m_0 \gamma}$

m_0 = rest mass
= mass when at rest
($v=0$)

and

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$E = 4$ GeV
$m_0 = m_\mu = 0.1$ GeV$/c^2$

$$4\,\cancel{\text{GeV}} = \left(0.1\ \frac{\cancel{\text{GeV}}}{\cancel{c^2}}\right)\gamma \cdot \cancel{c^2}$$

$$\Rightarrow \gamma = \frac{4}{0.1} = 40$$

$$\Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2} = \frac{1}{(40)^2}$$

$$\Rightarrow 1-\frac{1}{(40)^2} = \frac{v^2}{c^2} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{(40)^2}}$$

$$\simeq 1 - \frac{1}{2}\frac{1}{(40)^2} + \cdots$$

$$\simeq 1 - \frac{1}{3200}$$

$$\simeq 0.999687\ldots$$

Wrong answers i.e., without time dilation
(without Special Relativity)

$\Rightarrow$ $(2.2 \times 10^{-6}\ sec) \cdot (\text{speed of muon}) =$ (distance that muon can travel)

$$\simeq (2.2 \times 10^{-6}\ sec) \cdot (3 \times 10^{8}\ m/sec)$$

$$= 6.6 \times 10^{2}\ m$$

$$\ll 1.5 \times 10^{3}\ m$$

$\Rightarrow$ Cosmic muon can never reach the Earth.

Correct answer is that due to time dilation, the time duration seen by the observer on Earth is

$$(2.2 \times 10^{-6}\ sec) \cdot \gamma = (2.2 \times 10^{-6}\ sec) \cdot (40)$$

$$= 8.8 \times 10^{-5}\ sec$$

time dilation

$\Rightarrow$ The muon can travel

$$(40) \cdot (6.6 \times 10^{2}\ m) = 26.4 \times 10^{3}\ m$$

$$> 15 \times 10^{3}\ m$$

Length Contraction

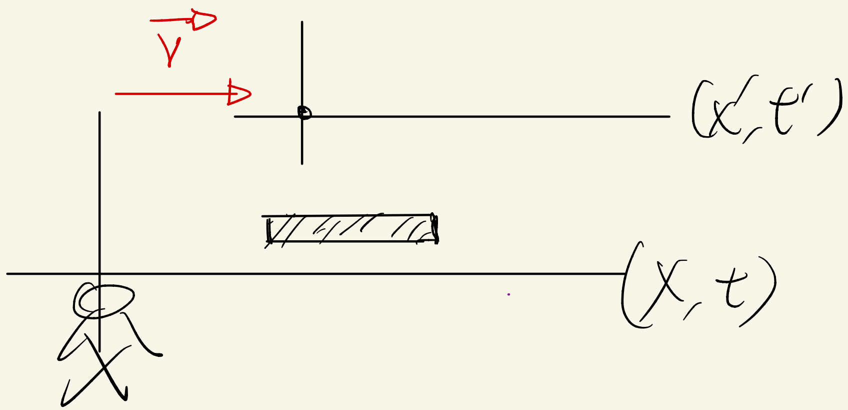
$$x = \gamma(x' + vt')$$

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

For $\Delta t' = 0$, $\Rightarrow \Delta x = \gamma \Delta x'$

$$\Rightarrow \boxed{\Delta x' = \frac{\Delta x}{\gamma}}, \quad \text{with } \gamma >> 1$$

Length contraction



From the viewpoint of the moving muon, the distance it travels through the atmosphere is shrunk (Lorentz length contraction) by $\gamma = 40$ (in this case).

Lorentz Velocity transformation

$$\boxed{\begin{aligned} x &= \gamma(x' + vt) \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}} \Rightarrow \boxed{\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) \end{aligned}}$$

Velocities

(divided by dt')

$$u_x \equiv \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}$$

(along the direction of $\vec{v}$)

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u_y'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$u_z = \frac{u_z'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

(Perpendicular to the moving direction of $\vec{v}$)

The inverse transformations can be readily written down by changing v to $-v$,

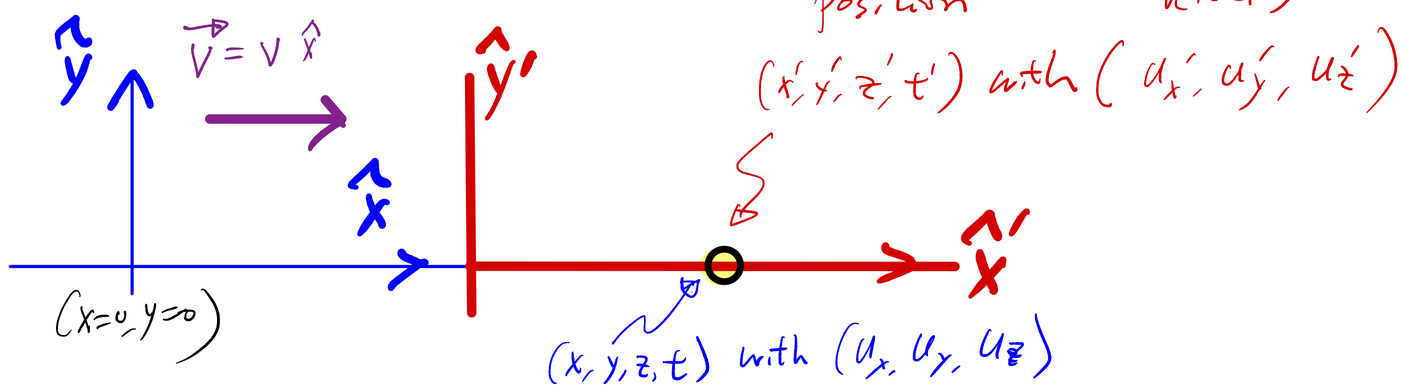
$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, \text{ etc.}$$

Note: No speed can exceed the speed of light ($c = 3\times10^8$ m/sec)

Example: $u_x' = c,\ v = c$

$$\Rightarrow u_x = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'} = \frac{c + c}{1 + \frac{c}{c^2}c} = c$$

Lorentz boost along $+\hat{x}$-direction



$$x = \gamma_V\left(x' + \frac{V}{c}t'\right)$$

$$t = \gamma_V\left(t' - \frac{V}{c^2}x^2\right)$$

$$y = y'$$

$$z = z'$$

$$u_x = \frac{u_x' + V}{1 + \frac{V}{c^2}u_x'}$$

$$u_y = \frac{u_y'}{\gamma_V\left(1 + \frac{V}{c^2}u_x'\right)}$$

$$u_z = \frac{u_z}{\gamma_V\left(1 + \frac{V}{c^2}u_x'\right)}$$

$$x' = \gamma_V\left(x - \frac{V}{c}t\right)$$

$$t' = \gamma_V\left(t - \frac{V}{c^2}x\right)$$

$$y' = y$$

$$z' = z$$

$$u_x' = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}$$

$$u_y' = \frac{u_y}{\gamma_V\left(1 - \frac{V}{c^2}u_x\right)}$$

$$u_z' = \frac{u_z}{\gamma_V\left(1 - \frac{V}{c^2}u_x\right)}$$

with

$$\boxed{\gamma_V = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}}$$