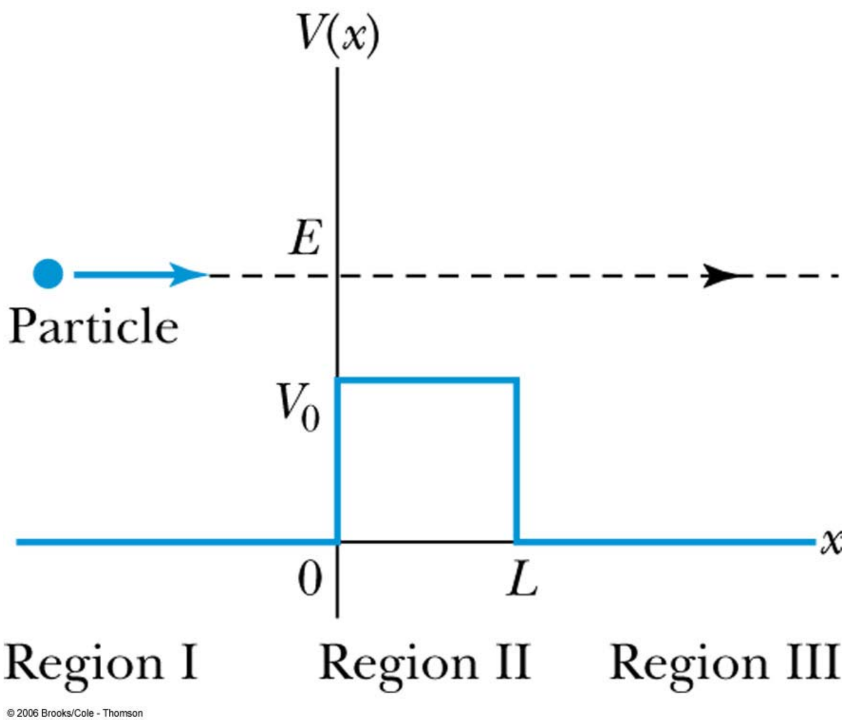
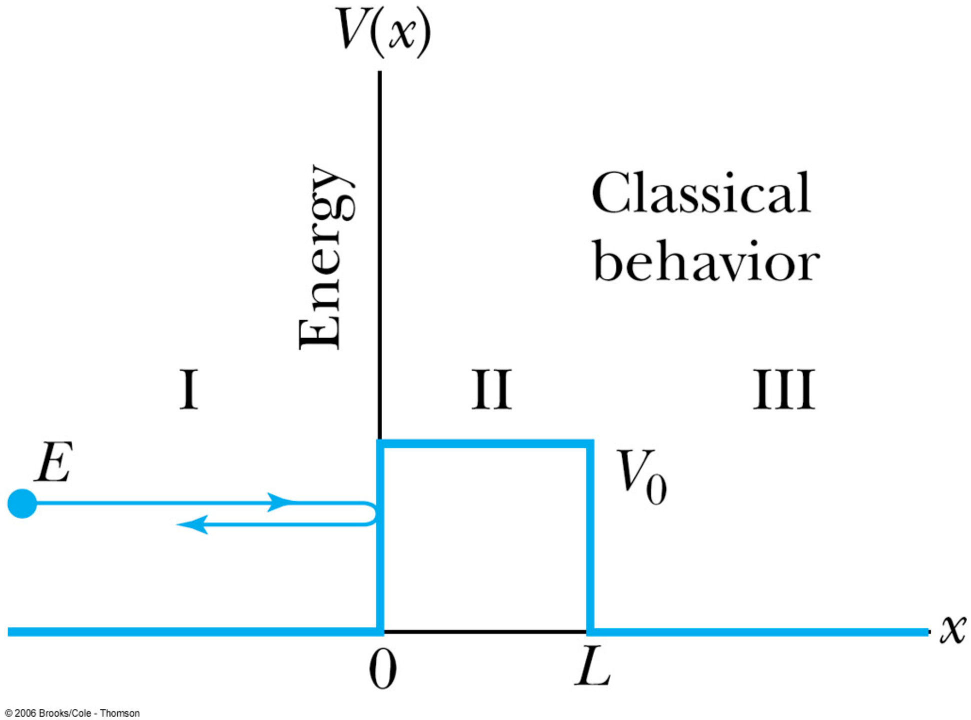
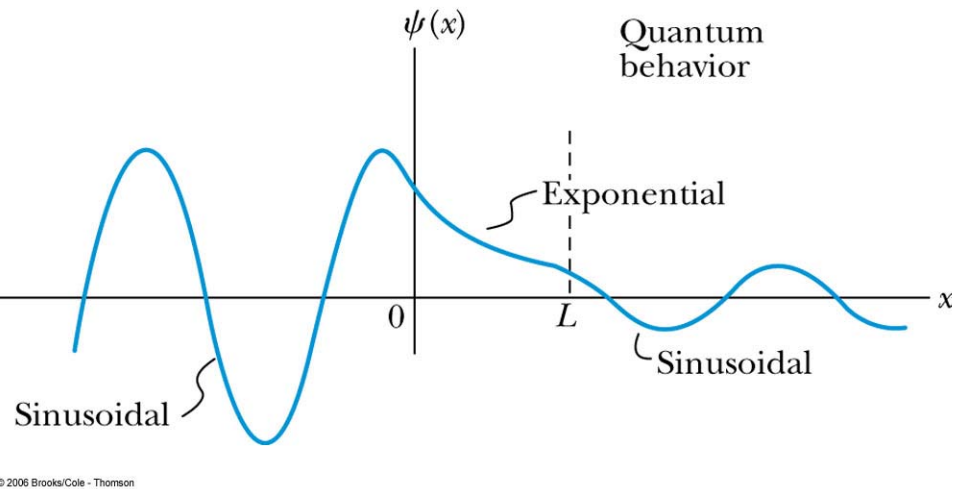
Tunneling





Classically, if a particle approaches a barrier with $E>V_0$, it is transmitted. If $E<V_0$, the classical particle will be reflected but the quantum particle can also tunnel through.

In Regions I,II and III, the Schroedinger equations are



$$
\begin{aligned}
\frac{d^2u_I}{dx^2} + \frac{2m}{\hbar^2}Eu_I &= 0 \\
\frac{d^2u_{II}}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)u_{II} &= 0 \\
\frac{d^2u_{III}}{dx^2} + \frac{2m}{\hbar^2}Eu_{III} &= 0.
\end{aligned}
$$

If we set

$$
k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}},
$$

the solutions are familiar:

$$\begin{aligned} u_I(x) &= Ae^{ikx} + Be^{-ikx} \\ u_{II}(x) &= Ce^{-\kappa x} + De^{\kappa x} \\ u_{III}(x) &= Fe^{ikx} + Ge^{-ikx}. \end{aligned}$$

Remember that e^{ikx} moves right and e^{-ikx} moves left. In region III, we can set G=0 because there is only a transmitted wave there.

At this point, we match solutions at $x = 0$ and $x = L$, using, for example,

$$\begin{aligned} u_{II}(L) &= u_{III}(L) \\ \frac{du_{II}(L)}{dx} &= \frac{du_{III}(L)}{dx} \end{aligned}$$

with a similar expression connecting u_I and u_{II} at $x = 0$. The interesting quantity is the ratio $|F|^2/|A|^2$ that measures the tunneling probability. Solving for F, one finds

$$\frac{F}{A} = \frac{2e^{-ika}}{[2\cosh(\kappa a) + i(\kappa/k - k/\kappa)\sinh(\kappa a)]}.$$

The tunneling probability is then

$$\frac{|F|^2}{|A|^2} = \left[1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)}\right]^{-1}.$$

When κL is large, this becomes

$$\frac{|F|^2}{|A|^2} \rightarrow 16\,\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L}.$$

Suppose an electron is accelerated through a 5 volt potential and strikes a 10 volt barrier of width 0.8 nm. What fraction of the electrons penetrate the barrier?

Here, L=0.8 nm, V_0=10 eV, E=5 eV and κ is

$$\begin{aligned}\kappa &= \frac{\sqrt{2m(V_0 - E)}}{\hbar} \\ &= \frac{\sqrt{2(0.511 \times 10^6\,\mathrm{eV/c^2})(10\,\mathrm{eV} - 5\,\mathrm{eV})}}{6.58 \times 10^{-16}\,\mathrm{eV\,s}} \\ \kappa &= \frac{3.43 \times 10^{18}\,\mathrm{s}^{-1}}{c} = 1.15 \times 10^{10}\,\mathrm{m}^{-1}.\end{aligned}$$

From this, κL=9.2, which is large compared to 1. We can then use

$$\frac{|F|^2}{|A|^2} = 16\left(\frac{5\,\mathrm{eV}}{10\,\mathrm{eV}}\right)\left(1 - \left(\frac{5\,\mathrm{eV}}{10\,\mathrm{eV}}\right)\right)e^{-18.4} = 4.1 \times 10^{-8}.$$