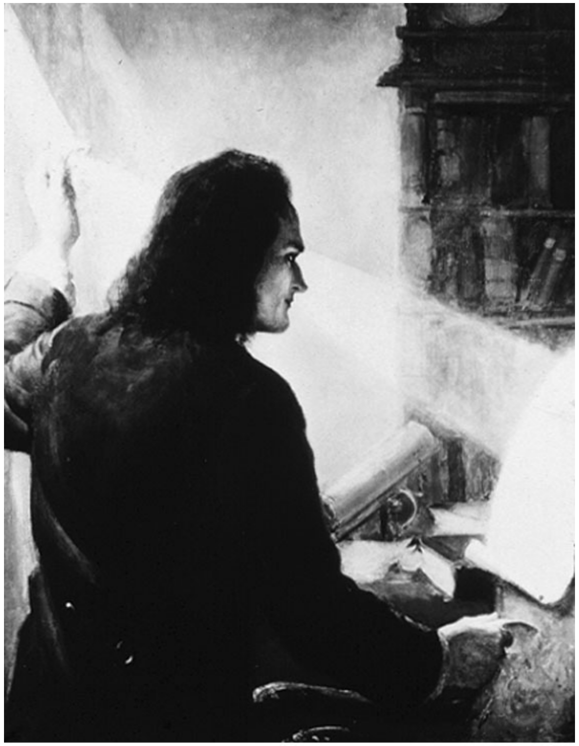
Special Relativity

PHY 215
Thermodynamics and
Modern Physics

Spring 2026
MSU



Newton's Laws

1. An object in motion with constant velocity will continue in its motion unless acted upon by some net external force.
2. The acceleration of a body is proportional to the force and inversely proportional to the mass of the body. Mathematically,

$$\vec{F} = m\vec{a}$$

3. The force exerted by body 1 on body 2 is equal in magnitude and opposite in direction to the force that body 2 exerts on body 1.

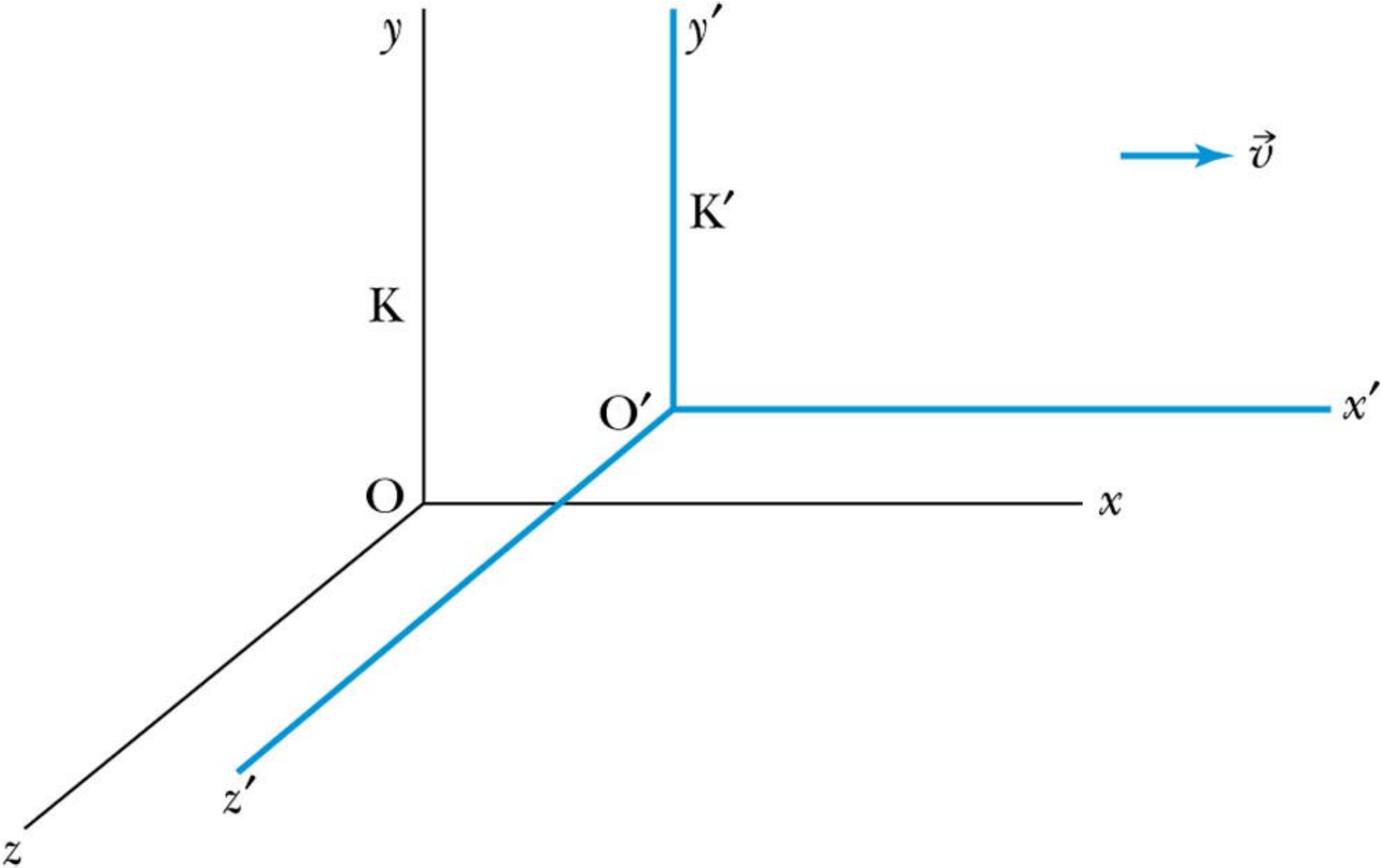
$$\vec{F}_{12} = -\vec{F}_{21}$$

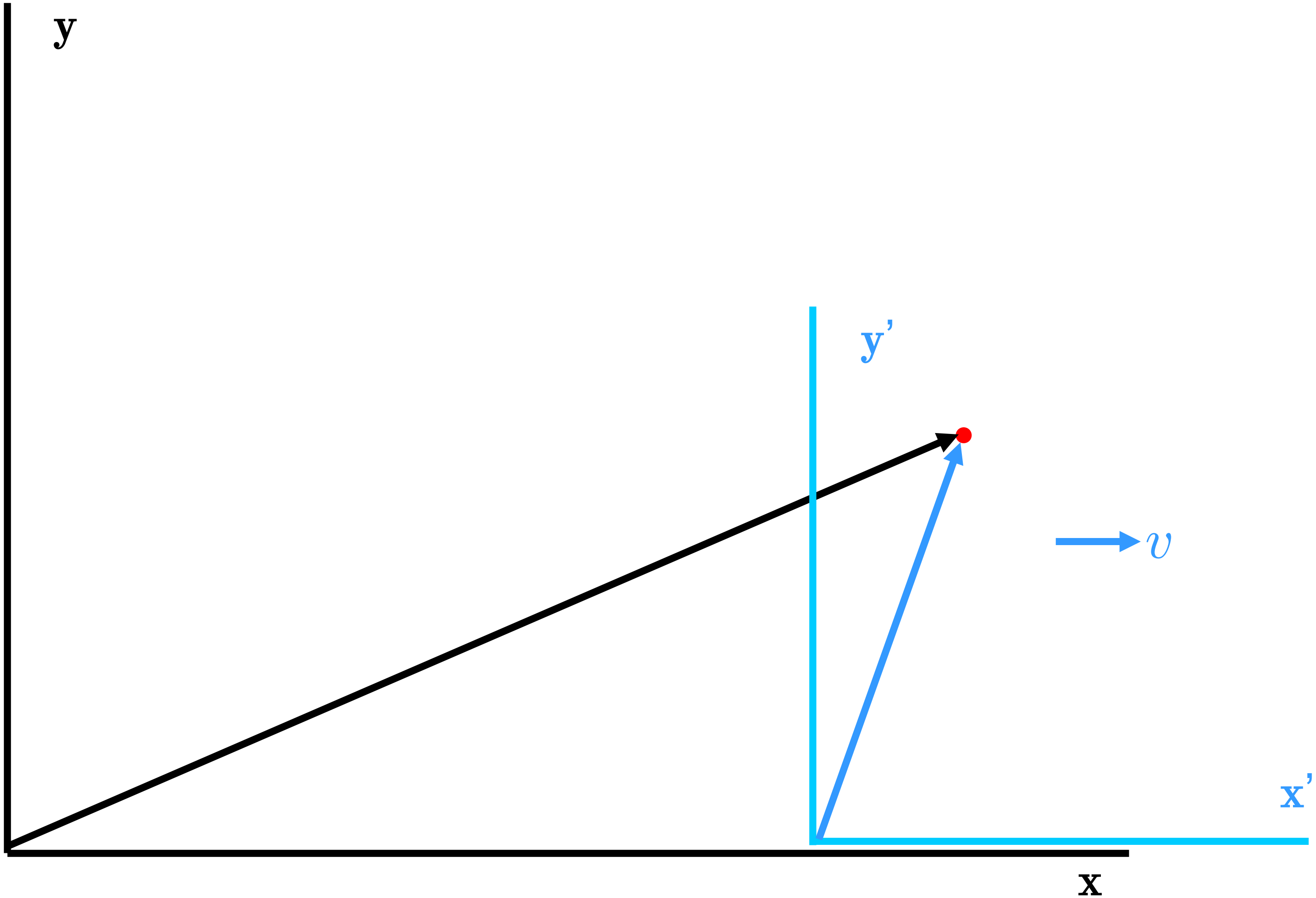
This is often called the *law of action and reaction.*

These laws can be used to derive the consequences of classical mechanics.

Special relativity represents a new kinematics that differs from Newtonian or non-relativistic kinematics, but must reduce to it under the right circumstances.

The 1st Law says something important about Newtonian mechanics: The dynamics doesn't change if one is at rest or moving with constant velocity.


y
y′
$\vec{v}$
K′
K
O′
x′
O
x
z′
z



y
y'
v
x'
x

The transformation that connects the coordinates in the primed and unprimed systems is

$$x' = x - vt, \; y' = y, \; z' = z, \; t' = t.$$

$\Rightarrow$ $\dot{x}' = \dot{x} - v, \; \dot{y}' = \dot{y}, \; \dot{z}' = \dot{z}$

It then follows that the accelerations in the two frames satisfy

$$\vec{a}' = (\ddot{x}', \ddot{y}', \ddot{z}') = (\ddot{x}, \ddot{y}, \ddot{z}) = \vec{a}$$

and $\vec{F} = m\vec{a}$ is unchanged.

These transformations have a nice property. If

$$x'' = x' - v't \quad x' = x - vt$$

then $x'' = x - (v + v')t = x - v''t.$

Consequently, successive transformations along the x-axis can be related to the initial transformation by merely adding the velocities. Obviously, if $v' = -v$, then $x'' = x$ and we are back to where we started.

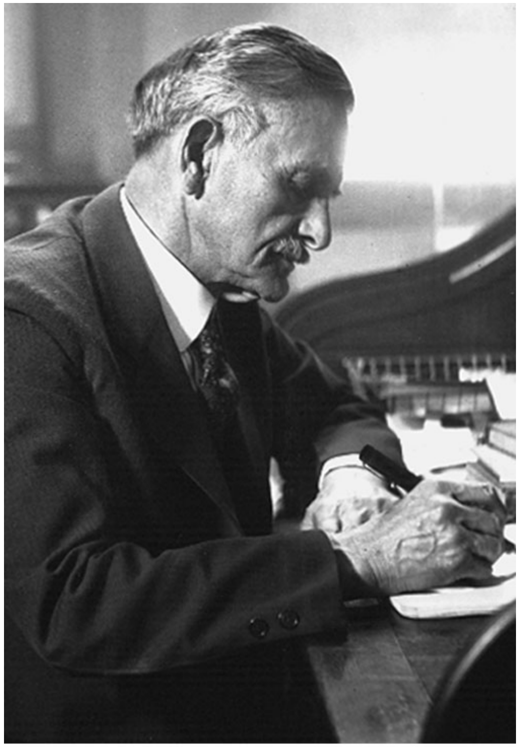
This type of behavior qualifies these symmetry transformations as a mathematical group that has come to be known as the Galilean group. Note that the behavior of the coordinate transformations depends on the equality $t' = t$.

<u>Newtonian or Galilean Relativity</u>

There is a notion of relative motion or Galilean relativity in classical mechanics and it relies on having a common time for both frames.

Search for the Ether

In 1860's, Maxwell showed that his theory of electromagnetism predicted light waves that propagated with a velocity c which could be calculated. Since all known waves propagated in a medium, the medium for light propagation, called the *ether*, was of particular interest.



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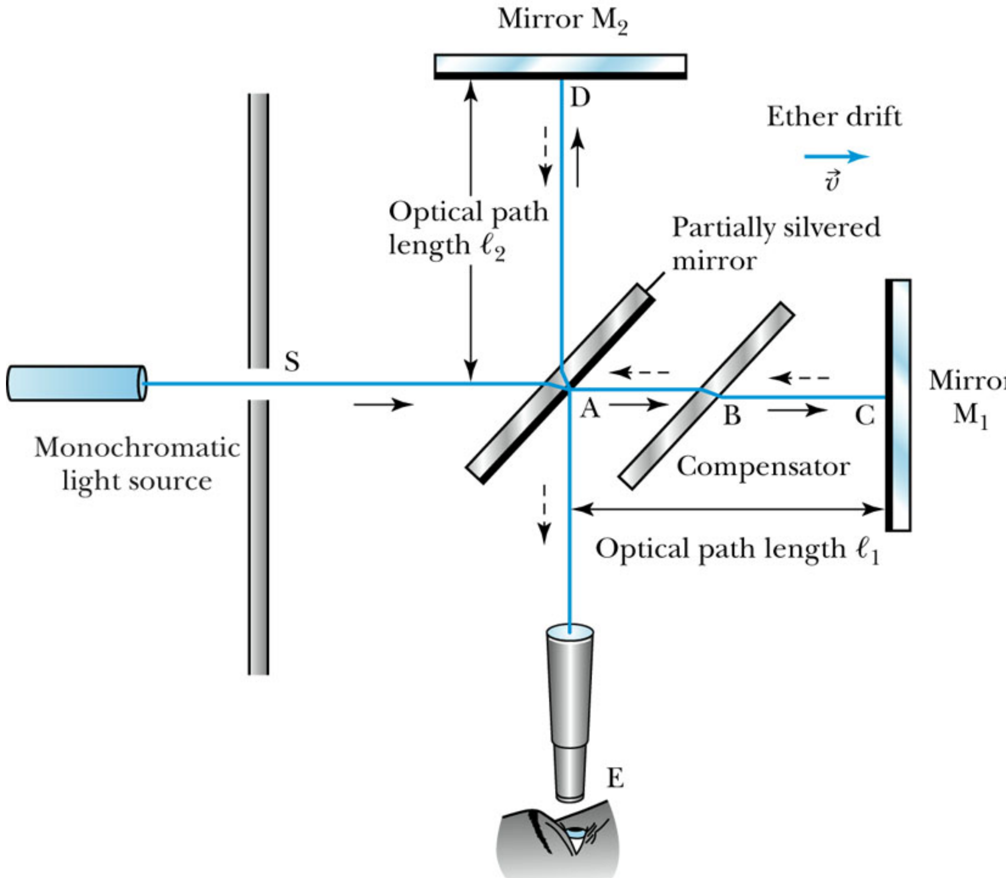
Albert Michelson (1852-1931) devised an experiment to detect the (stationary) ether by using optical interference.



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For the light transit with or against the ether velocity, the light has velocity $c+v$ or $c-v$, and the total time taken is

$$t_1 = \frac{\ell_1}{c+v} + \frac{\ell_1}{c-v} = \frac{2\ell_1}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

When the light moves perpendicular to the Earth's velocity, its velocity is $\sqrt{c^2-v^2}$ so

$$t_2 = \frac{2\ell_2}{\sqrt{c^2-v^2}} = \frac{2\ell_2}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

The time difference for this configuration is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{\ell_2}{\sqrt{1-v^2/c^2}} - \frac{\ell_1}{1-v^2/c^2}\right)$$

Upon rotation by 90°, the time difference is

$$\Delta t' = t'_2 - t'_1 = \frac{2}{c}\left(\frac{\ell_2}{1-v^2/c^2} - \frac{\ell_1}{\sqrt{1-v^2/c^2}}\right)$$

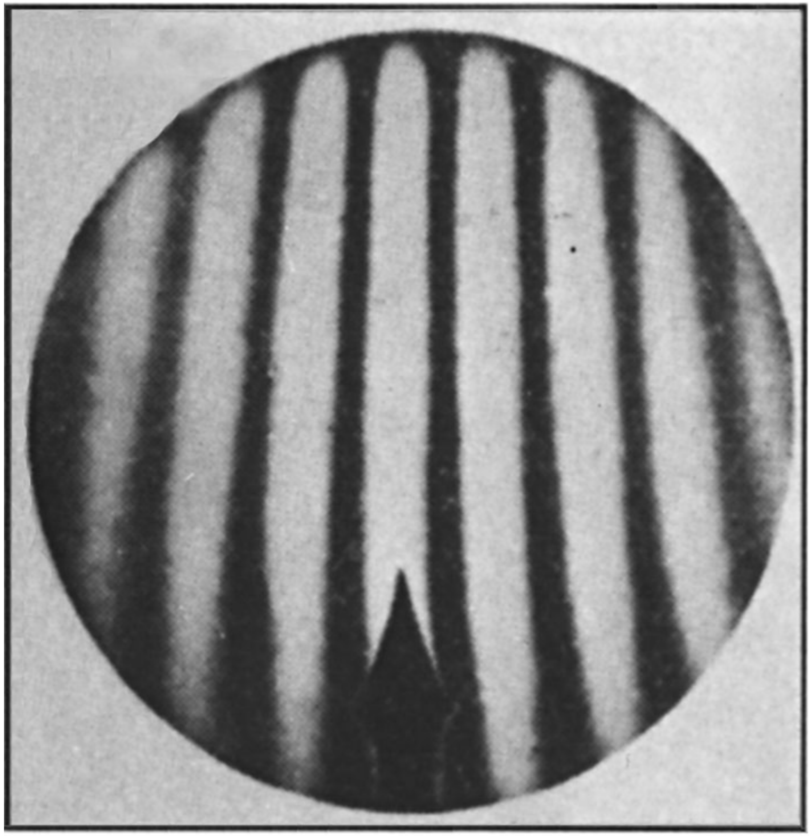
The time delay between the two configurations is then

$$\Delta t' - \Delta t = \frac{2}{c}\left(\frac{\ell_1 + \ell_2}{1 - v^2/c^2} - \frac{\ell_1 + \ell_2}{\sqrt{1 - v^2/c^2}}\right)$$

If this is expanded in terms of v^2/c^2

$$\Delta t' - \Delta t = \frac{2}{c}(\ell_1 + \ell_2)\left(\frac{v^2}{c^2} - \frac{v^2}{2c^2}\right) = \frac{v^2(\ell_1 + \ell_2)}{c^3}$$

Using the Earth's velocity around the Sun as $3\text{x}10^4$ m/s, Michelson calculated a time difference of $8\text{x}10^{-17}$ s that amounted to 0.04 of a fringe, something he could detect.



The experiment was performed and no fringe shift was detected. It was later repeated in Europe with the same null result.

A third measurement using an apparatus with much longer arms was performed at Case University with Edward Morley, again giving a null result reported in 1887.

The Michelson-Morley result is clearly a failure of the Galilean transformation and the significance of this work was recognized when Michelson was awarded the Nobel prize in 1907.

This inability to detect the ether was a serious problem. Many arguments were advanced to explain the null result. The most radical of these was proposed by FitzGerald and Lorentz – the longitudinal beam was somehow contracted by a factor

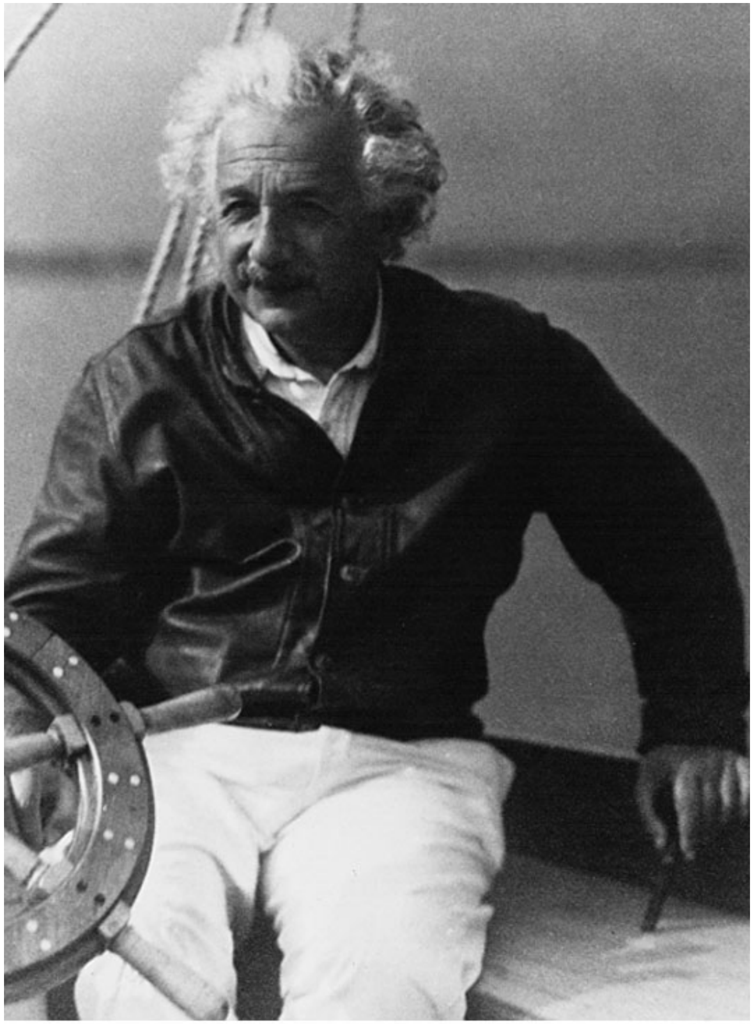
$$\sqrt{1 - v^2/c^2}.$$

This makes the time difference vanish

$$\Delta t' - \Delta t = \frac{2}{c}\left(\frac{(\ell_1 + \ell_2)\sqrt{1 - v^2/c^2}}{1 - v^2/c^2} - \frac{\ell_1 + \ell_2}{\sqrt{1 - v^2/c^2}}\right)$$

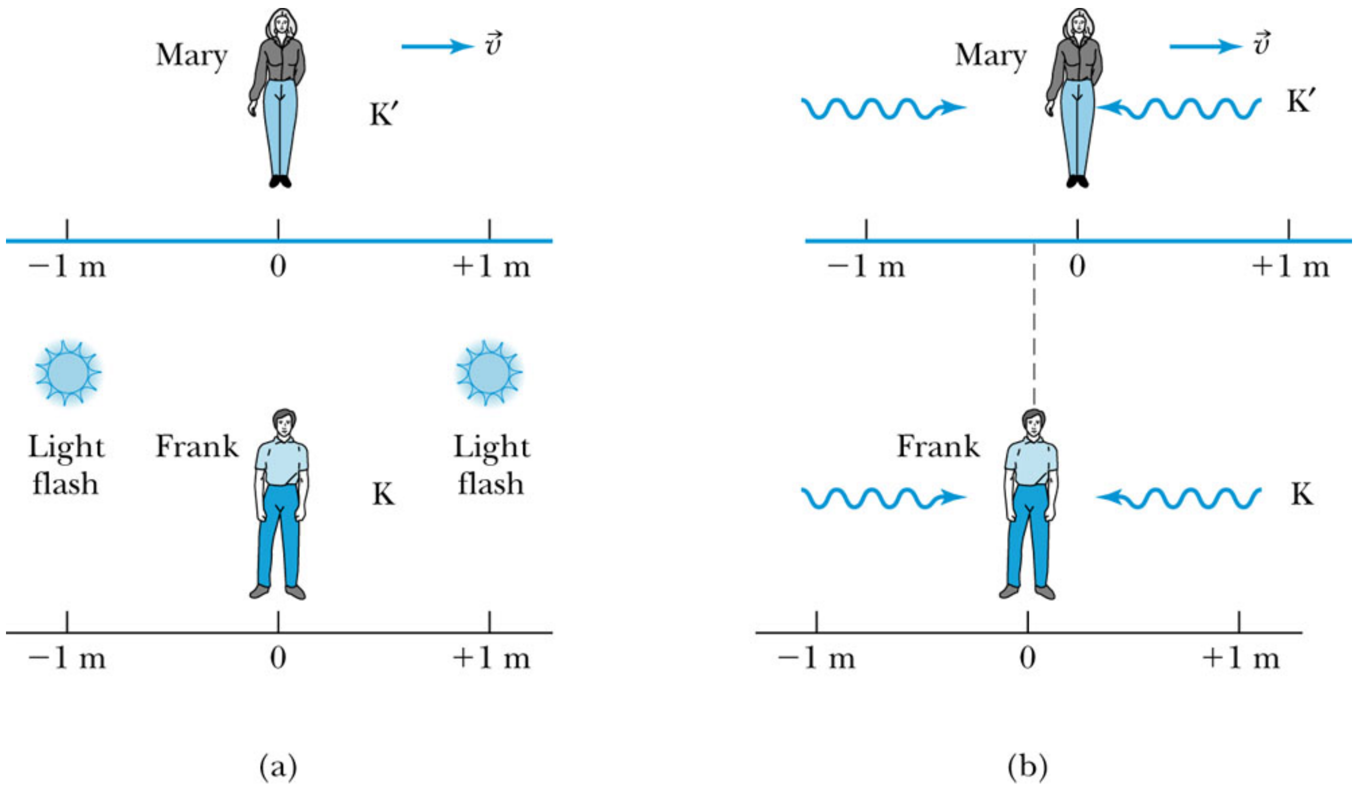
but seems *ad hoc*. However, it turns out to be the correct idea.

Special Relativity



Some 20 years after Michelson's null experiment, Einstein, who claimed he was unaware of this result, reexamined the transformation rules with particular attention to the behavior of light.

He reasoned that if the velocity of light were a fixed constant then time should transform because results judged to be simultaneous in a stationary frame needn't be so in a moving frame.



Reconciling this observation led Einstein to propose two postulates that are the foundation on special relativity:

1. <u>The Principle of Relativity:</u> The laws of physics are the same in all inertial frames. There is no way to detect absolute motion, and no preferred inertial system exists.

2. <u>The constancy of the speed of light:</u> Observers in all inertial systems measure the same value for the speed of light in a vacuum.

According to 2., if frames K and K′ have their origins coincident at t=t′=0, and a light pulse is emitted, both frames will see a spherical wave with

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

The important thing about this set of conditions is that t and t′ are not equal. What we need is a relation between the primed and unprimed variables that preserves these equations.

Section 2.4 of the text goes through the steps of finding the necessary relations. For our purposes, let's write down the result and check that it works.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

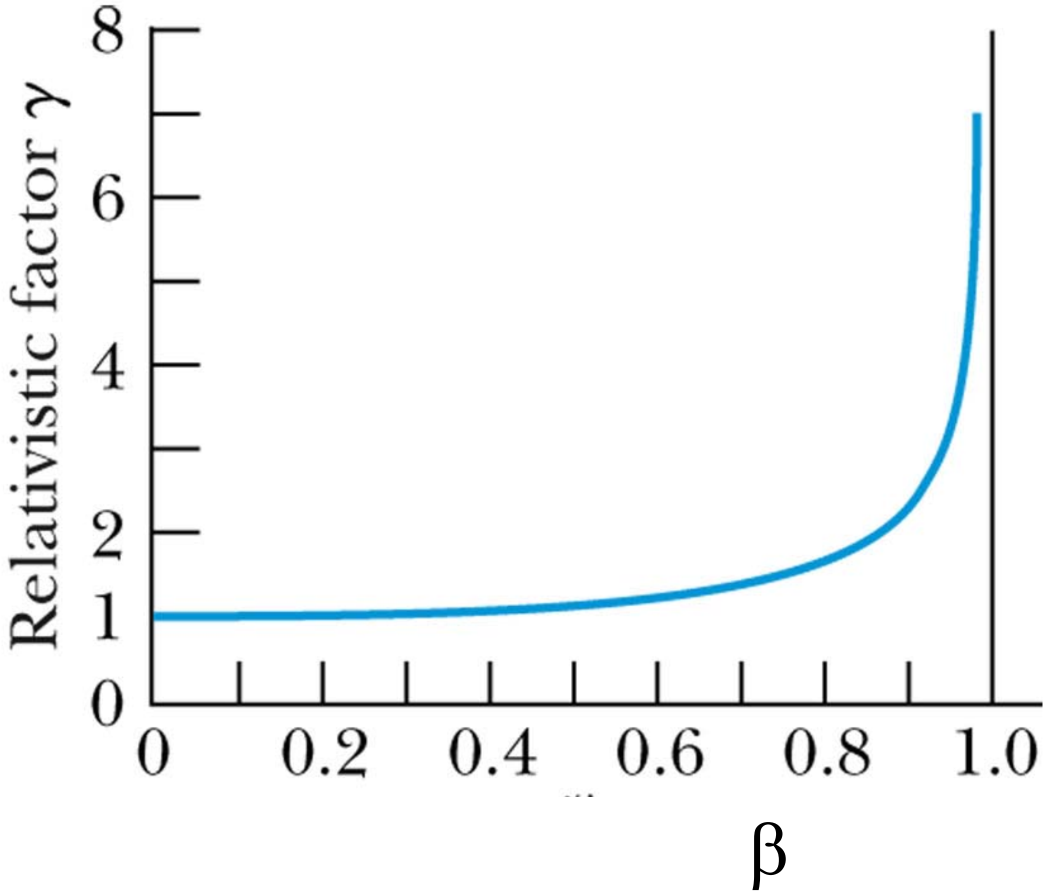
These are the Lorentz transformations.

First, note that under ordinary circumstances $v << c$ and these relations reduce to the Galilean transformation.

Second, for v a substantial fraction of c, the factor

$$\frac{1}{\sqrt{1 - v^2/c^2}}$$

becomes large.



$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}; \beta = \frac{v}{c}$$

As a check,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = (\gamma(x - \beta ct))^2 + y^2 + z^2 - c^2(\gamma(t - \beta x/c))^2$$

$$= \gamma^2(1 - \beta^2)(x^2 - c^2t^2) + y^2 + z^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

The quantity

$$x^2 + y^2 + z^2 - c^2t^2$$

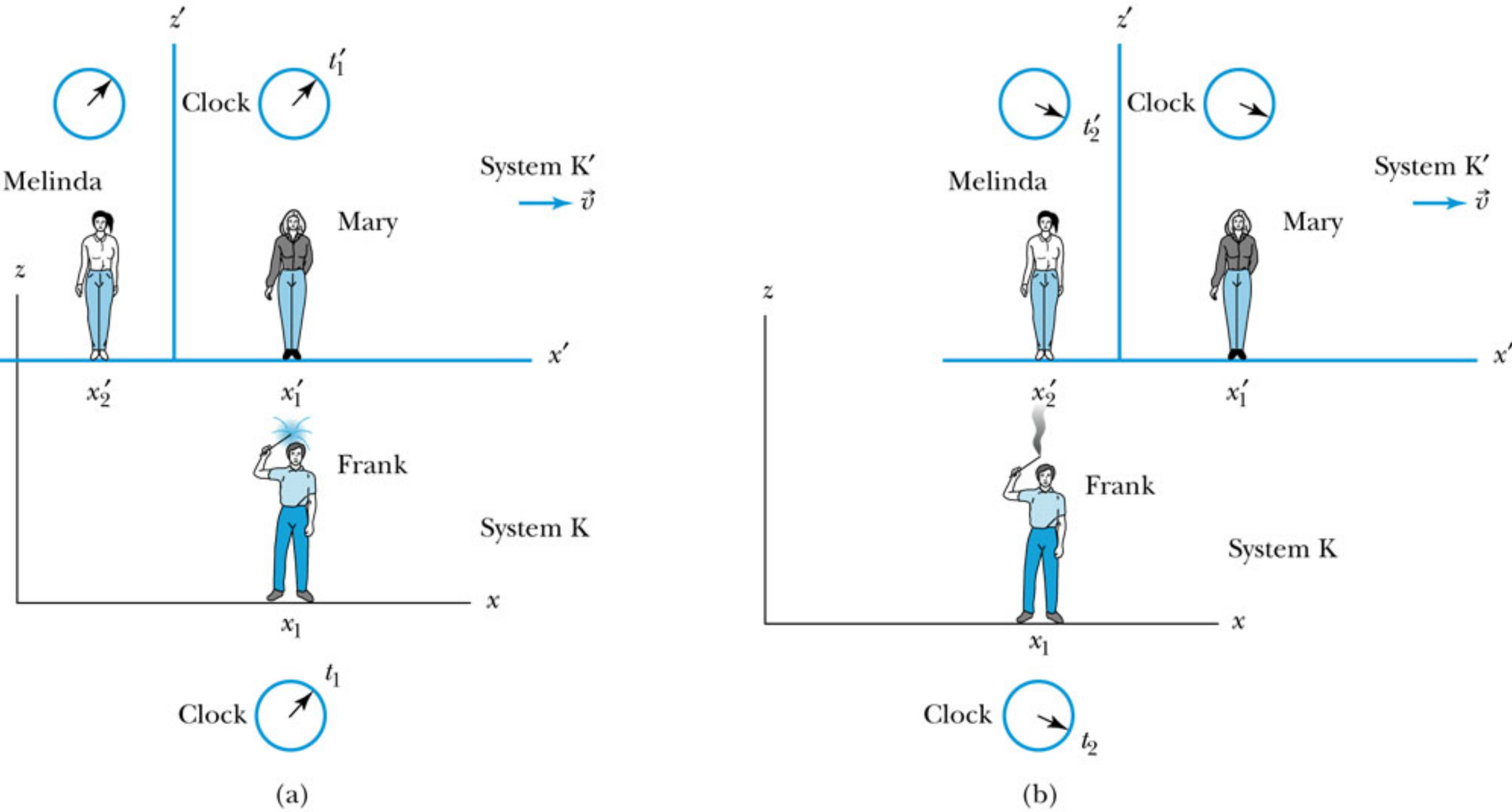
is called an invariant of the theory. It has the <u>same</u> value when calculated in any system.

Consequences of Special Relativity

We now imagine two systems K and K′. Each has its own measuring devices and synchronized clocks.

Time Dilation:

In the unprimed system, Frank lights a sparkler a holds it until it goes out. In the primed system, Mary observes the start of the burn and Melinda observes the end.



z′
Clock
t'_1
Melinda
System K′
$\vec{v}$
Mary
z
x′
x'_2
x'_1
Frank
System K
x
x_1
Clock
t_1
(a)
z′
Clock
t'_2
Melinda
System K′
$\vec{v}$
Mary
z
x′
x'_2
x'_1
Frank
System K
x
x_1
Clock
t_2
(b)

The burn lasts $t_2 - t_1$ in system K. To find $t'_2 - t'_1$ in K', use

$$t'_2 - t'_1 = \frac{(t_2 - t_1) - v(x_2 - x_1)/c^2}{\sqrt{1 - v^2/c^2}}$$

Since $x_2 = x_1$, the time interval in K', T', is

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

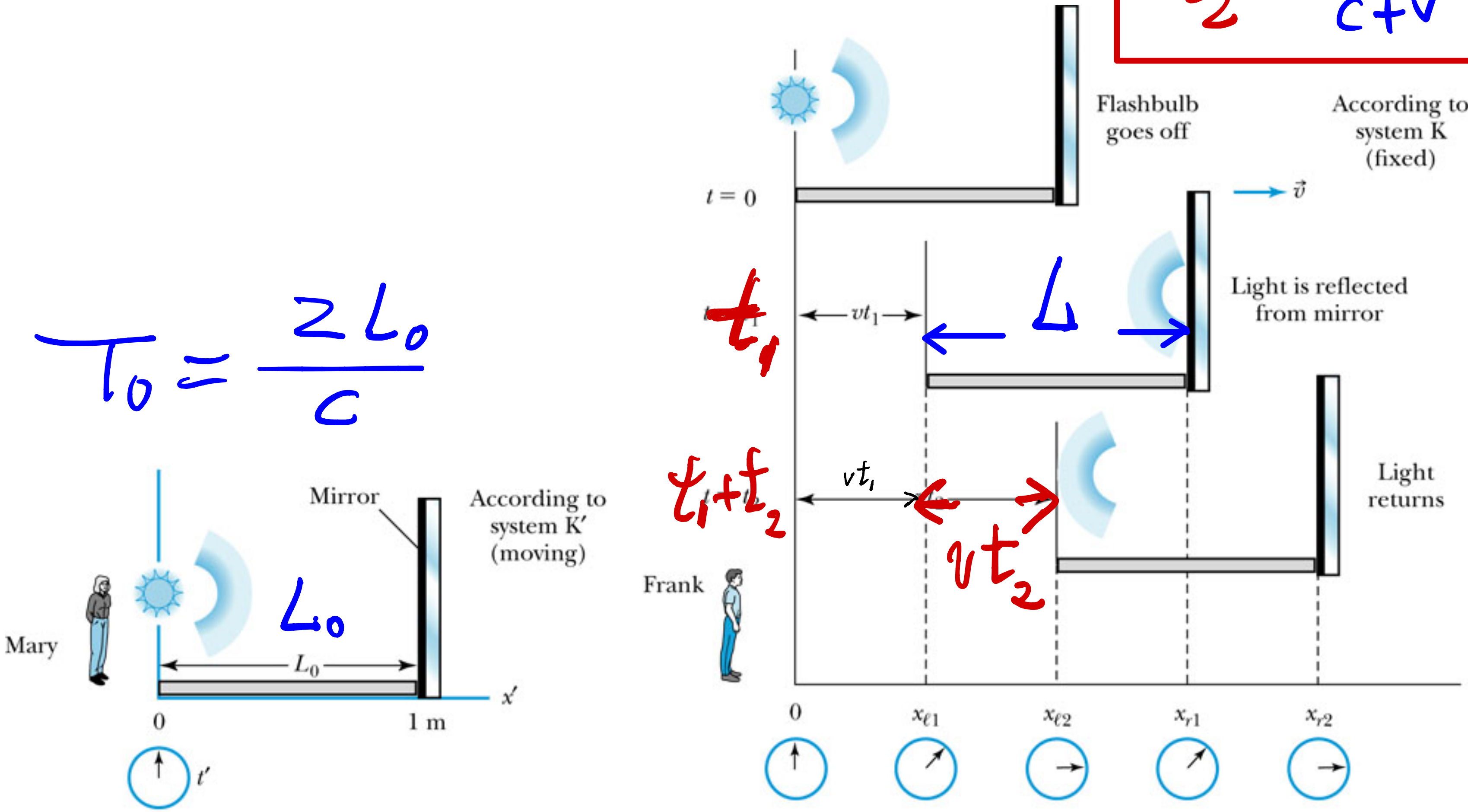
where $T' = t'_2 - t'_1$ and the proper time is $T_0 = t_2 - t_1$.

Length Contraction:

$$t_1 = \frac{L}{c-v}$$

$$t_2 = \frac{L}{c+v}$$

$$T_0 = \frac{2L_0}{c}$$



(a)

(b)

On the way to the mirror, the light pulse travels

$$L + vt_1 = ct_1 \quad \Rightarrow t_1 = \frac{L}{c-v}$$

and, after reflection, it reaches the left end of the rod after traveling

$$L - vt_2 = ct_2 \quad \Rightarrow t_2 = \frac{L}{c+v}$$

The total time for the round trip is

(1) $$t_1 + t_2 = T = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c(1-v^2/c^2)}$$

But, $\mathcal{T}$ is related to the proper time T_0 as

(2) $$\mathcal{T} = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \frac{2L_0}{c\sqrt{1 - v^2/c^2}},$$ where $T_0 = \frac{2L_0}{c}$

Comparing these expressions gives

(1) $\mathcal{T} = \frac{2L}{c\left(1 - \frac{v^2}{c^2}\right)}$

$$L = \sqrt{1 - v^2/c^2}\,L_0 = \frac{L_0}{\gamma}, \quad \gamma \geq 1$$

This is just the FitzGerald-Lorentz contraction factor that was suggested to give a null result for the Michelson-Morley experiment.

Review

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, y' = y, z' = z, t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

$$L = \sqrt{1 - v^2/c^2}L_0 = \frac{1}{\gamma}L_0$$

Addition of Velocities

The addition of velocities presents a challenge in special relativity since the Lorentz transformations imply that $v<c$.

We can calculate the velocities in the K′ system by differentiating with respect to t′. For x′, the Lorentz transformation gives

$$u'_x = \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt}\frac{dt}{dt'} - v\frac{dt}{dt'}\right) = \gamma\left(\frac{u_x - v}{dt'/dt}\right)$$

To calculate the denominator, we can use the relation between t′ and t

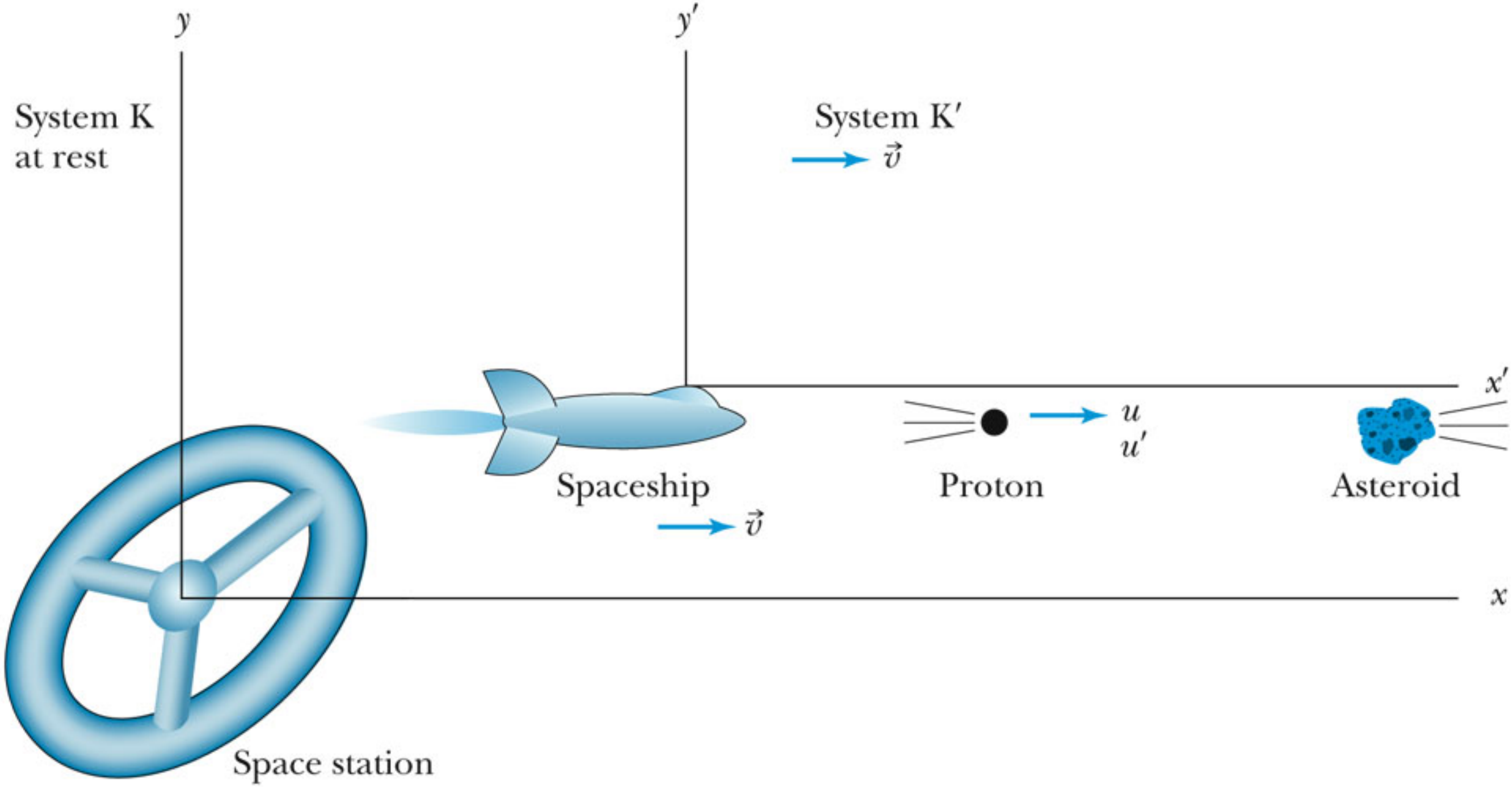
$$\frac{dt'}{dt} = \gamma\left(1 - \frac{dx}{dt}v/c^2\right) = \gamma\left(1 - u_x v/c^2\right).$$

The expression for u'_x is

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}$$

This may be inverted to give

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$



y
y′
System K
at rest
System K′
$\vec{v}$
x′
u
u′
Spaceship
Proton
Asteroid
$\vec{v}$
x
Space station

$$v = 0.6c,\ u'_x = 0.99c$$

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$

$$= \frac{0.99c + 0.6c}{1 + 0.99 \times 0.6}$$

$$u_x = 0.997c$$

Unlike the transverse components y and z, that are unchanged for motion in the x-direction, the transverse velocities do change.

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = \frac{u_y}{dt'/dt}$$

$$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}$$

Similarly,

$$u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)}$$

Suppose that the spaceship has the same velocity 0.6c, but fires at an object moving in the y' direction. What velocity does someone on the space station see?

Solution

Here we need to calculate γ.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.6^2}} = 1.25$$

The two components of the velocity are

$$u_x = \frac{0 + v}{1 + 0 \times v/c^2} = v = 0.6c$$

$$u_y = \frac{0.99c}{1.25(1 + 0.6 \times 0)} = 0.792c$$

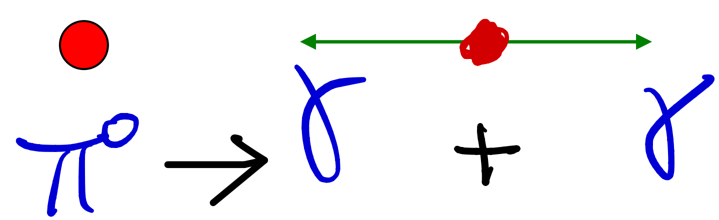
The magnitude of the velocity as seen in the space ship is then

$$
\begin{aligned}
u &= \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{0.6^2 + 0.792^2 + 0^2}c \\
&= 0.994c
\end{aligned}
$$

Tests of Special Relativity

In elementary particle physics, most particles are unstable and decay into stable final states. As a simple example, the π^0 meson decays into two photons. The natural lifetime in the π^0 system is 8.4×10^{-14} s.

In their rest system, the decay photons are back-to-back and each has velocity c.



The π^0's are produced in high energy collisions and have a velocity v when observed in the lab. According to the relativistic velocity transformation the velocity u in this frame is

$$u = \frac{c+v}{1+vc/c^2} = c$$

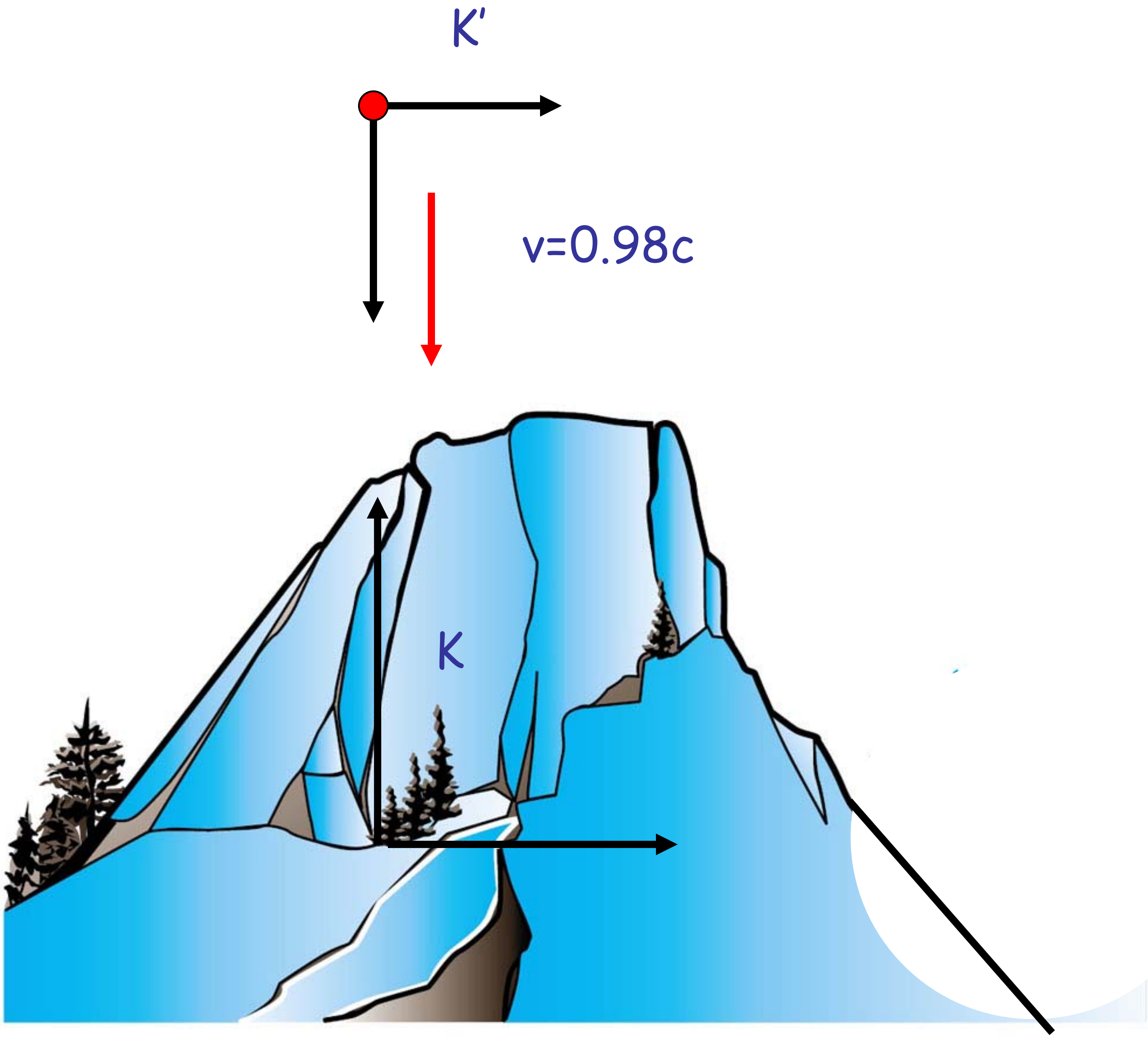
Thus, in an actual experiment, the constancy of the velocity of light in various frames can be verified.

Muon Decay

When cosmic rays strike the upper atmosphere, charge pions are produced. These decay predominantly into muons (μ) that are unstable. In their rest system, they decay according to the exponential law

$$N(t') = N(0)e^{-0.693t'/(t_{1/2})}$$

N(t') and N(0) are the number of muons at t'=0 and t'=t'. The constant $t_{1/2}$ is the half life of the muon, $t_{1/2} = 1.52\times10^{-6}$ s.


K'
v=0.98c
K

If 1000 muons are detected on a 2000 m mountain top, how many will be detected at ground level if the muons travel at 0.98 c?

The muons decay in their rest frame, where the time is t′ and the location is x′=0. In terms of the unprimed system, $t=\gamma t'$, so our equation in the Earth system is

$$N(t) = N(0)e^{-0.693t/(\gamma t_{1/2})}$$

$$= N(0)e^{-t/\gamma\tau_0}$$

mean

τ_0 = lifetime

In this system, t is just

$$t = \frac{2000m}{0.98c} = 6.80 \times 10^{-6}s$$

Since γ in this case is 5, the number of muons observed at ground level is

$$\begin{aligned} N &= 1000e^{-(0.693\times 6.80\ 10^{-6})/(5.0\times 1.52\times 10^{-6})} \\ &= 540 \end{aligned}$$

From the muon's point of view, the mountain looks shorter. It is contracted by

$$L' = \frac{L}{\gamma} = 400\ m$$

and is coming toward the muon at 0.98c. The muon calculates the time to reach the ground as

$$t' = \frac{400m}{0.98c} = 1.36 \times 10^{-6} s$$

Now, using $t_{1/2} = 1.52 \times 10^{-6}$ s, the number of muons arriving at ground level is, again, 540.

$$N = 1000 \cdot e^{-0.693\, t'/(t_{1/2})}$$

$$= 540$$

$$\ln(2) = 0.693$$