Special Relativity

PHY 215
Thermodynamics and
Modern Physics

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MSU

Relativistic Momentum

We have determined how coordinates and velocities transform under the constraints of special relativity. Einstein's assertion that all physical processes are the indistinguishable in inertial frames implies that we need to figure out how Newton's mechanics fits into this scheme.

Newton's 2nd Law can be written

$$\frac{d\vec{p}}{dt} = \vec{F}, \ \vec{p} = m\vec{u} = m\frac{d\vec{r}}{dt}.$$

In Galilean relativity, the relation between momenta in frames moving with relative velocity v along the x axis is

$$p'_x = p_x - v,\ p'_y = p_y,\ p'_z = p_z.$$

To retain the latter two relations, the definition of the momentum must be more complicated than $p_y = mu_y$ since we know that

$$mu'_y = \frac{mu_y}{\gamma(1 - vu_x/c^2)},$$

and, in general, these are not the same.

Now, we want the momentum to be in the direction of the velocity, so it will look something like

$$\vec{p} = mf(u/c)\vec{u}, \ u = \sqrt{u_x^2 + u_y^2 + u_z^2}.$$

Then, the equality of the y-components of the momentum means

$$mf(u')u'_y = mf(u)u_y.$$

The choice of f(u) must be such that f(u') just cancels the extra factors in going from u'_y to u_y and reproduces f(u). The f(u) that does the job is

$$f(u) = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

The transformation rule is

$$\frac{1}{\sqrt{1 - u'^2/c^2}} = \frac{(1 - vu_x/c^2)}{\sqrt{1 - \beta^2}} \frac{1}{\sqrt{1 - u^2/c^2}}.$$

Thus, the definition

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$$

ensures that $p'_y = p_y$ and $p'_z = p_z$. What happens to the relation between p'_x and p_x ?

$$\begin{aligned} p'_x &= \frac{m u'_x}{\sqrt{1 - u'^2/c^2}} \\ &= \frac{m\cancel{(1 - vu_x/c^2)}}{\sqrt{1 - \beta^2}} \frac{1}{\sqrt{1 - u^2/c^2}} \frac{(u_x - v)}{\cancel{(1 - vu_x/c^2)}} \end{aligned}$$

or

$$p'_x = \gamma\left(\frac{mu_x}{\sqrt{1-u^2/c^2}} - \frac{mv}{\sqrt{1-u^2/c^2}}\right)$$

$$= \gamma\left(p_x - \frac{mc^2}{\sqrt{1-u^2/c^2}}\frac{v}{c^2}\right)$$

In terms of the variable cp_x,

$$cp'_x = \gamma\left(cp_x - \beta\frac{mc^2}{\sqrt{1-u^2/c^2}}\right)$$

This form shows that the quantity

$$\frac{mc^2}{\sqrt{1-u^2/c^2}}$$

enters the transformation of cp_x in the same way ct enters the transformation of x.

$$x' = \gamma\left(x - \beta ct\right).$$

Furthermore, this expression has dimensions of energy, so let's write

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}.$$

To interpret this expression, assume $u<<c$ and expand.

$$E \simeq mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right) = mc^2 + \frac{1}{2}mu^2 + \ldots.$$

The right side is a constant, mc^2, and the Newtonian expression for the kinetic energy. E is called the total energy consisting of the rest energy, mc^2, and the relativistic kinetic energy, K, defined as

$$K = E - mc^2.$$

The total energy E must transform like ct, so

$$E' = \gamma \left(E - \beta c p_x \right).$$

To summarize, momentum and total energy are

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}, \quad E = \frac{mc^2}{\sqrt{1-u^2/c^2}}.$$

From these equations, a number of useful relations follow

a. $$\frac{\vec{u}}{c} = \frac{c\vec{p}}{E}$$

$$b. \quad c^2\vec{p}^{\,2} + m^2c^4 = \frac{m^2c^2\vec{u}^{\,2}}{(1-u^2/c^2)} + m^2c^4$$
$$= \frac{m^2c^4}{(1-u^2/c^2)} = E^2.$$

More simply,

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

Energy Units

A very common energy unit is derived from the kinetic energy acquired when a charge q is accelerated by an voltage V. The kinetic energy of an electron accelerated by a potential difference of 1 volt is called an electron volt (eV).

$$W = (1e)(1V) = (1.602 \times 10^{-19} C)(1V)$$
$$1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}.$$

For example, the rest energy of a proton is

$$
\begin{aligned}
m_p c^2 &= (1.672 \times 10^{-27}\,\mathrm{kg})(3 \times 10^{8}\,\mathrm{m/s})^2 \\
&= 1.50 \times 10^{-10}\,\mathrm{J}\,\frac{1\,\mathrm{eV}}{1.602 \times 10^{19}\,\mathrm{J}} \\
&= 9.38 \times 10^{8}\,\mathrm{eV} = 938\,\mathrm{MeV}.
\end{aligned}
$$

In these units, the proton mass is

$$m_p = 938\,\mathrm{MeV}/c^2.$$

We often use $m_p = 1$ GeV.

Example

A proton has a kinetic energy of 5 GeV. What are its total energy, momentum and velocity?

$$\begin{aligned} a.\ E &= K + mc^2 = 5000\,\text{MeV} + 938\,\text{MeV} \\ &= 5938\,\text{MeV}. \end{aligned}$$

$$\begin{aligned} b.\ pc &= \sqrt{E^2 - m^2c^4} \\ &= \sqrt{(5938)^2 - (938)^2}\,\text{MeV} \\ p &= 5863\,\text{MeV/c}. \end{aligned}$$

$$c.\ \frac{u}{c} = \frac{pc}{E} = \frac{5863}{5938} = 0.987.$$

Further discussion – massless particles

We have established two important relations between velocity, momentum and energy.

$$E = \sqrt{p^2c^2 + m^2c^4} \text{ and } \frac{u}{c} = \frac{pc}{E}.$$

What happens if m approaches 0?

From the energy relation

$$E \to \sqrt{p^2c^2 + 0} = pc,$$

and from the relation determining u

$$\frac{u}{c} = \frac{pc}{pc} = 1.$$

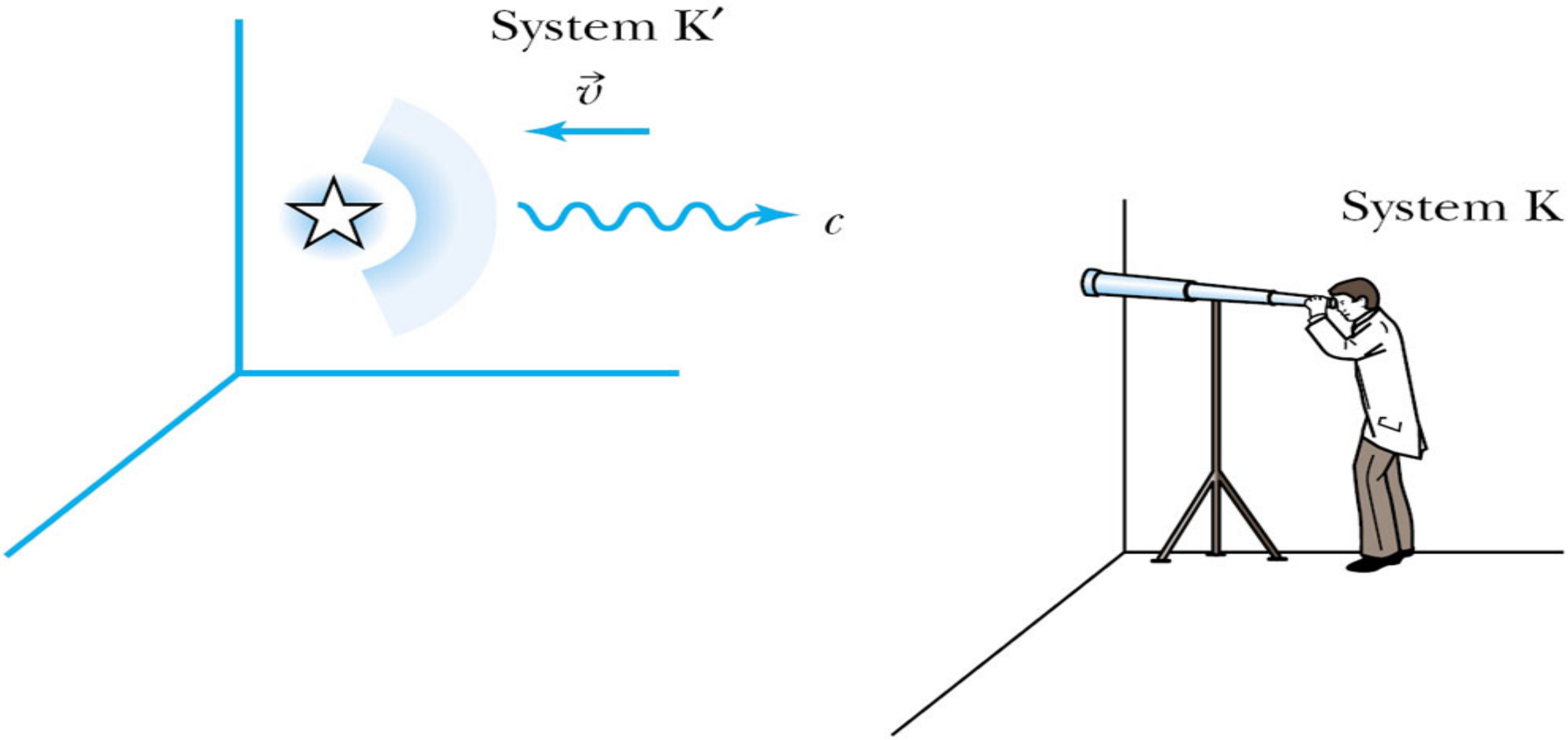
Hence, we learn that massless particles travel at the velocity of light.

Moreover, the energy and momentum of a massless particle are related as

$$p = \frac{E}{c}.$$

The primary example of a massless particle is the photon. This is not accidental. The theory of electromagnetism contains a symmetry that requires the photon to be massless.

Doppler Shift



Suppose and observer in system K views light from a star that is receding with velocity v. In a time T, the length of the wave train is

$$L = cT + vT = (c + v)T.$$

If there are n waves, the wavelength is

$$\lambda = \frac{(c + v)T}{n},$$

and the frequency f is

$$f = \frac{c}{\lambda} = \frac{nc}{(c+v)T}.$$

In the K' system of the star, the number of waves is $n = f_0 T_0$, and the relation between T and T_0 is

$$T = \gamma T_0.$$

Using this to solve for n gives the frequency change observed in system K.

$$f = \frac{f_0 cT/\gamma}{(c+v)T} = \frac{\sqrt{1-\beta^2}}{(1+\beta)} f_0$$

$$f = \sqrt{\frac{1-\beta}{1+\beta}} f_0.$$

Since $\lambda f = c$, the observed wavelength increases to compensate for the decrease in the frequency and the light is red shifted.

Example:

The H_α line of hydrogen in a star has a wavelength of 656.4×10^{-9} m. What wavelength will be observed on Earth if the star is receding at $\beta = 0.5$?

Solution:

$$\lambda = \frac{c}{f} = \sqrt{\frac{1+\beta}{1-\beta}}\frac{c}{f_0}$$

$$\lambda = \sqrt{\frac{1+\beta}{1-\beta}}\lambda_0.$$

Putting in the numbers

$$\lambda = \sqrt{\frac{1.5}{0.5}} 656.4 \times 10^{-9}\,\mathrm{m}$$

$$\lambda = 1137. \times 10^{-9}\,\mathrm{m}.$$

Four vectors

We have seen that the expression

$$c^2t^2 - x^2 - y^2 - z^2$$

is invariant under the Lorentz transformations

$$x' = \gamma(x - \beta ct),\ y' = y,\ z' = z,\ ct' = \gamma(ct - \beta x).$$

What is interesting is that the expression

$$\frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2$$

is also invariant under the same Lorentz transformation, since

$$\frac{E'^2}{c^2} - p_x'^2 - p_y'^2 - p_z'^2 = \gamma^2\left(\frac{E}{c} - \beta p_x\right)^2 - \gamma^2\left(p_x - \beta\frac{E}{c}\right)^2 - p_y^2 - p_z^2$$

$$= \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2$$

There is a convenient way of expressing this invariance. We can introduce a generalization of the ordinary vector of Newtonian mechanics that is called a 4 vector. For position and momentum, these are defined by

$$\begin{aligned} x^\mu &= (ct, x, y, z) = (ct, \vec{r}) \\ p^\mu &= (\frac{E}{c}, p_x, p_y, p_z) = (\frac{E}{c}, \vec{p}). \end{aligned}$$

The scalar or dot product is defined as

$$x \cdot x = c^2 t^2 - x^2 - y^2 - z^2 = c^2 t^2 - \vec{r}^{\,2}$$

$$p \cdot p = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - \vec{p}^{\,2}.$$

Using this notation, the statement that a 4 vector is invariant under Lorentz transformation reduces to

$$x' \cdot x' = x \cdot x, \; p' \cdot p' = p \cdot p.$$

Turning the argument around, we can use the invariance of this scalar product to derive the Lorentz transformation.

Because of the sign difference between the time contribution and the space contribution, the scalar product of x^μ with itself can be positive, negative or zero.

a. For $x \cdot x > 0$, x is said to be timelike.

b. For $x \cdot x < 0$, x is said to be spacelike.

c. For $x \cdot x = 0$, x is said to be lightlike.

Finally, since x^μ and p^μ are 4 vectors, their scalar product is also invariant.

$$x' \cdot p' = x \cdot p.$$

From the forms of x^μ and p^μ, this is

$$t'E' - \vec{r'} \cdot \vec{p'} = tE - \vec{r} \cdot \vec{p}.$$

More 4 vectors

From the form of p^μ, we have

$$p \cdot p = E^2/c^2 - \vec{p}^2 = m^2c^2$$

so the momentum 4 vector for a particle of mass m is timelike. As the mass of the particle goes to zero, E goes to

$$E \to |\vec{p}|c.$$

It then follows that

$$p^\mu = (|\vec{p}|, \vec{p})$$
$$p \cdot p = 0.$$

We can then conclude that the momentum 4 vectors for physical particles satisfy

$$p \cdot p \geq 0.$$

Example: Particle Collisions

The Tevatron at Fermilab studies the collisions of protons and antiprotons with energies of 1 TeV and opposite momenta. The 4 momenta are

$$p^\mu = (E/c, \vec{p}), \; \bar{p}^\mu = (E/c, -\vec{p}).$$

Their total momentum squared is

$$(p + \bar{p}) \cdot (p + \bar{p}) = 4E^2/c^2 = 4\,\mathrm{TeV}^2/c^2.$$

(with $E = 1$ TeV)

A collision with this much momentum can probe a hypothetical particle of high mass (M). If this particle is produced at rest, its momentum will be

$$P^\mu = (Mc, \vec{0}), \qquad p^\mu + \bar{p}^\mu = P^\mu$$

Momentum conservation then gives

$$P \cdot P = M^2 c^2 = \frac{(Mc^2)^2}{c^2} = 4\,\frac{\mathrm{TeV}^2}{c^2}$$

$$\Rightarrow \quad Mc^2 \simeq 2\,\text{TeV}.$$

As you may remember from mechanics, head-on collisions of equal masses are the most efficient way to transmit energy because the center of mass is at rest. The Tevatron makes use of this idea to optimize its ‘mass reach’.

The advantage of the Tevatron over an accelerator that collides antiprotons with a stationary proton can be seen as follows.

In this case, the momentum 4 vectors are

$$p^\mu = (mc, \vec{0}),\ \bar{p}^\mu = (E/c, \vec{p}),$$

and

$$\begin{aligned}(p+\bar{p})\cdot(p+\bar{p}) &= (E/c + mc)^2 - \vec{p}^{\,2} \\ &= E^2/c^2 - \vec{p}^{\,2} + 2mE + m^2c^2 \\ (p+\bar{p})\cdot(p+\bar{p}) &= 2m(E + mc^2).\end{aligned}$$

If this is to equal the Tevatron's 4 Tev^2/c^2

$$(E+mc^2) \simeq E = \frac{4\,\text{TeV}^2}{2mc^2} = \frac{4\,\text{TeV}^2}{2\,\text{GeV}} = 2000\,\text{TeV}.$$

$(E \gg mc^2)$, Proton mass = 1 GeV/c^2

Space-Time Events

If two events occur at space-time points x_1^μ and x_2^μ, their space-time interval between them is

$$\begin{aligned}\Delta s^\mu &= x_2^\mu - x_1^\mu \\ &= (ct_2 - ct_1, x_2 - x_1, y_2 - y_1, z_2 - z_1).\end{aligned}$$

The scalar product

$$\Delta s \cdot \Delta s = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is Lorentz invariant.

$$\Delta s' \cdot \Delta s' = \Delta s \cdot \Delta s$$

As an example, imagine an event in which a person stands at the origin and turns on a light bulb for a time T. In this case,

$$\Delta s = (cT, 0, 0, 0).$$

In a frame moving along the x-axis with speed v, $\Delta s'$ is

$$\Delta s' = (cT', x_2' - x_1', 0, 0).$$

The invariance of $(\Delta s)^2$ then gives

$$c^2 T'^2 - (x_2' - x_1')^2 = c^2 T^2.$$

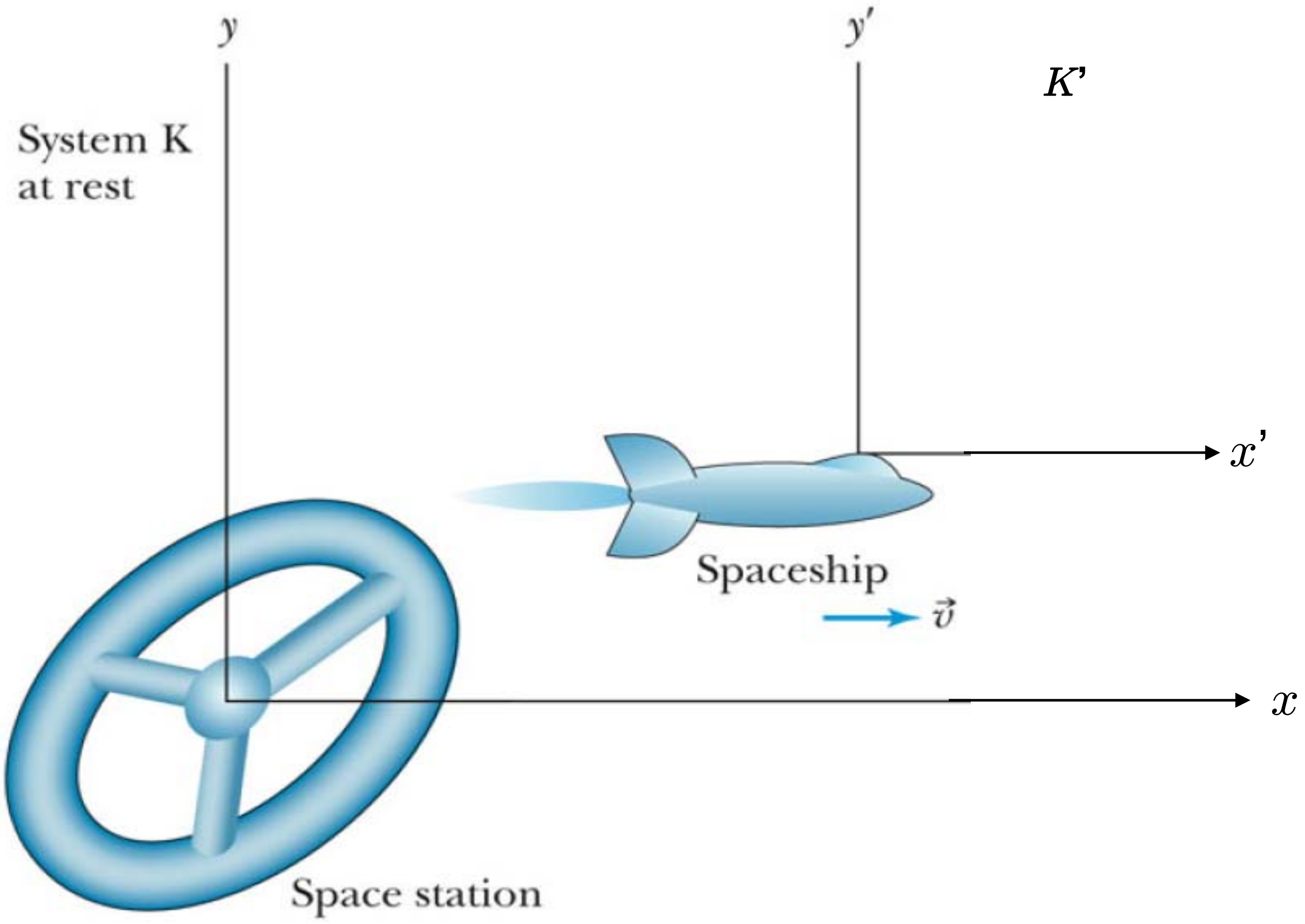
From the Lorentz transformation

$$(x_2' - x_1') = \gamma \beta c T,$$

so

$$\begin{aligned} c^2 T'^2 &= (1 + \beta^2 \gamma^2) c^2 T^2 \\ T' &= \gamma T. \end{aligned}$$

Twin Paradox



Mary travels to a star 8 light years from the stationary space platform K. Mary's space ship travels with velocity $v = 0.8c$. Her twin, Frank, observes the trip and calculates that one leg of the trip takes

$$T = \frac{8\,\text{ly}}{0.8c} = 10\,\text{years.}$$

Mary departs when she and Frank are 30, so Frank reckons he'll be 50 when she returns.

Since Mary is traveling at $v = 0.8c$, Frank knows that her clock is ticking slower by the factor

$$\frac{1}{\gamma} = \sqrt{1 - 0.8^2} = \frac{3}{5}.$$

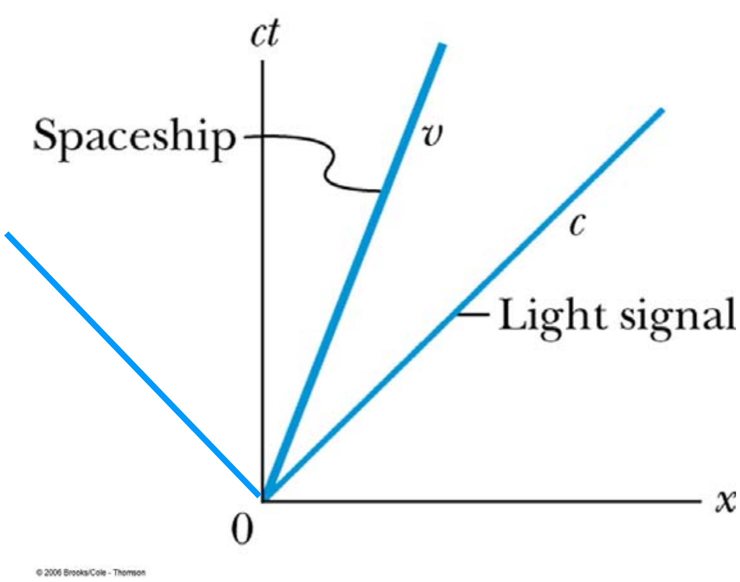
Hence, he calculates her the time to reach the star to be

$$T' = \frac{1}{\gamma}\left(\frac{8\text{ ly}}{0.8c}\right) = 6\text{ years}.$$

Consequently, Frank calculates Mary's age upon return as 42.

Note that, on the trip to the star, each twin believes the other is aging less rapidly. The asymmetry occurs because Mary must accelerate to begin the return trip.

World lines



In the x-ct plane, we can plot the trajectories of objects. Light signals follow a 45 degree line since $\Delta s^2 = 0$.

For an object traveling with a speed $v < c$,

$$x = vt = \beta ct$$

$$ct = x/\beta$$

which gives a line with a steeper slope.

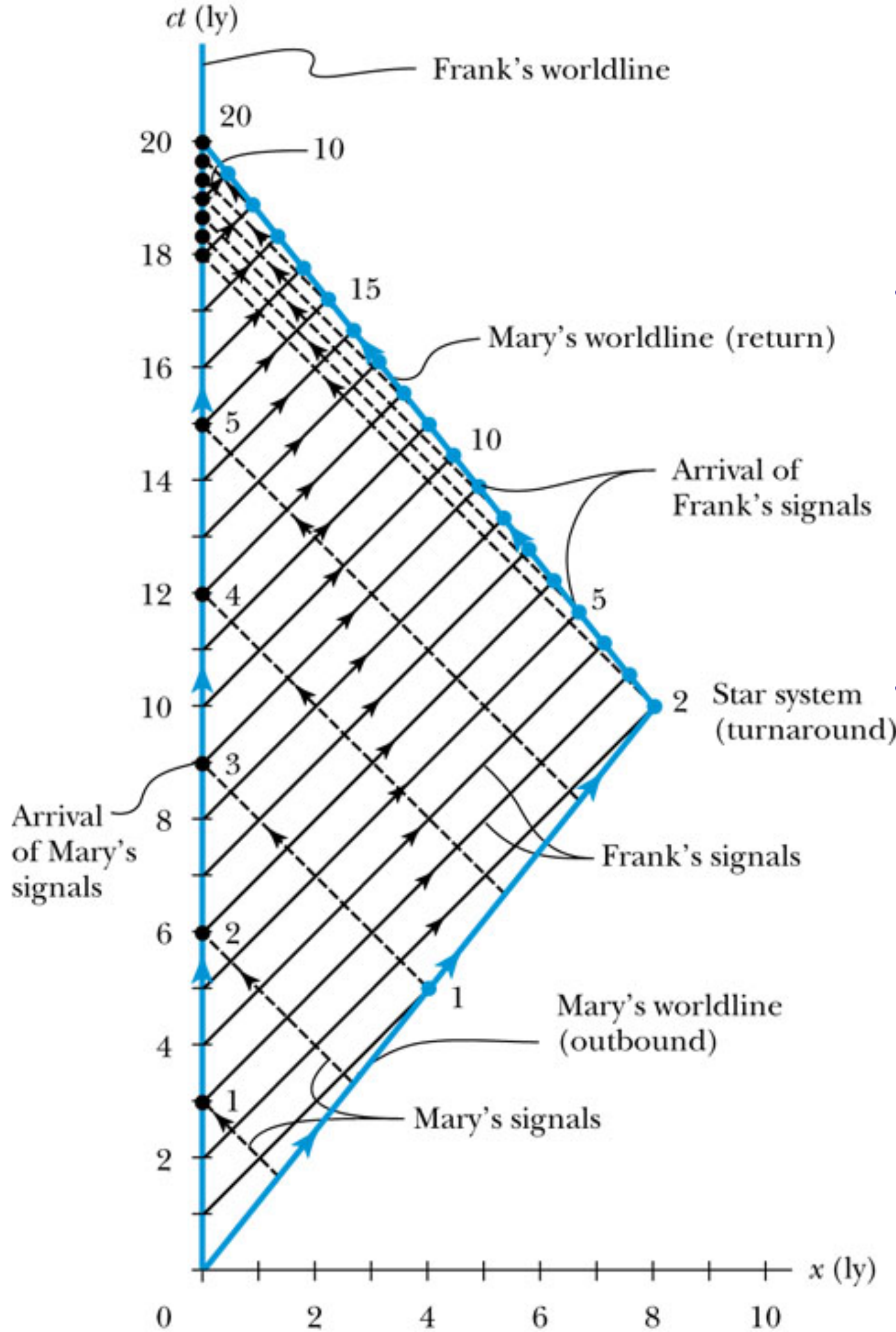
We can revisit the twin paradox using world lines and the Doppler shift. Suppose Mary and Frank agree to send a light pulse to each other every year according to their clock. The once/year frequency f_0 is shifted because of the space ship's motion.

On the trip to the star, the frequency shift is

$$f = \sqrt{\frac{1-0.8}{1+0.8}} f_0 = \frac{f_0}{3}$$

and on the way back the shift is

$$f = \sqrt{\frac{1+0.8}{1-0.8}} f_0 = 3 f_0$$



Now, on Mary's outward trip signals take 3 years to reach Frank. Frank receives last of the 6 signals from the Mary's outward trip in year 18. He then receives 6 signals in the 2 years before Mary reaches the station, concluding that she has aged 12 years.

Only 2 Frank's signals reach Mary before she turns back. Then, on the return trip, she receives 3 signals/year for 6 years. Mary counts 20 signals and concludes that Frank has aged 20 years.