The Bohr model of the Hydrogen atom

- Certain stationary states exist, they are stable.
- The energies of the states are well defined. When a transition happens, electromagnetic radiation is emitted / absorbed. The frequency of the radiation:

  $$E_2 - E_1 = \Delta E = hf$$

- The angular momentum of a state is quantized:

  $$L = mvr = n \cdot \hbar \quad \text{where } \hbar = \frac{h}{2\pi}$$

  $n = 1, 2, 3, \ldots$

  n: principal quantum number

  (Originally: $K = \frac{1}{2}mv^2 = \frac{1}{2} nh f_{orb}$

  $$mv^2 = nh \frac{v}{2\pi r}$$

  $$mvr = n \frac{h}{2\pi} = n\hbar$$)

Successes of the Bohr model

Size of the H-atom:

$$r_n = n^2 \cdot a_0 = n^2 \cdot \frac{4\pi\varepsilon_0 \hbar^2}{me^2}$$

$a_0 = 0.529\,\text{Å}$: Bohr radius.

Energies of the states:

$$E_n = -\frac{E_0}{n^2} = -\frac{me^4}{2\hbar^2 (4\pi\varepsilon_0)^2} \cdot \frac{1}{n^2}$$

$E_0 = 13.6\,\text{eV}$: ionization energy of the H-atom.

Rydberg constant:

$$R_\infty = \frac{E_0}{hc} = \frac{me^4}{4\pi c \hbar^3 (4\pi\varepsilon_0)^2}$$

$$R_\infty = 1.097373 \cdot 10^7 \; 1/\text{m}$$

$$R_H = 1.096776 \cdot 10^7 \; 1/\text{m}$$

Electron velocity:

$$\beta_n = \frac{v_n}{c} = \frac{\alpha}{n} = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \cdot \frac{1}{n}$$

$\alpha \approx \frac{1}{137}$: fine structure constant

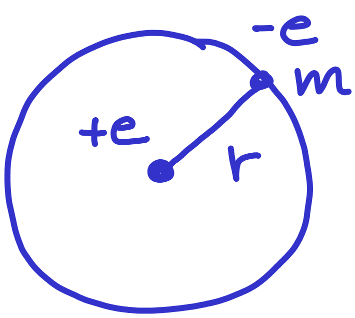
Bohr's Correspondance Principle

In the limit of large quantum numbers, quantum mechanics reduces to classical mechanics.

$$n = \underbrace{1, 2, 3,}_{\text{quantum}} \ldots \quad \underbrace{\infty}_{\text{classical}}$$

mechanics

Radii of orbits



Newton's second law with electric force:

$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{r^2} = m\,\frac{v^2}{r}$$

$$\frac{e^2}{4\pi\varepsilon_0 m r} = v^2$$

Bohr condition:

$$L = mvr = n\hbar$$

$$v = \frac{n\hbar}{mr}$$

$$v^2 = \frac{n^2\hbar^2}{m^2 r^2}$$

Let's combine them:

$$\frac{e^2}{4\pi\varepsilon_0 m r} = \frac{n^2\hbar^2}{m^2\cdot r^2}$$

$$r = \frac{4\pi\varepsilon_0\hbar^2}{m e^2}\cdot n^2 = n^2\cdot\underbrace{a_0}_{0.529\,\mathring{A}}$$

$$r = \frac{4\pi\varepsilon_0\hbar^2}{m e^2 Z}\cdot n^2 = n^2\cdot\frac{a_0}{Z}$$

Energies of orbits

$$E = KE + PE = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} =$$

$$= \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r}$$

$$E_n = -\frac{1}{2}\cdot\frac{e^2}{4\pi\varepsilon_0 n^2 a_0} = -\frac{1}{n^2}\cdot\frac{e^2}{8\pi\varepsilon_0 a_0} =$$

$$= -\frac{1}{n^2}\cdot\frac{e^2}{8\pi\varepsilon_0 \frac{4\pi\varepsilon_0\hbar^2}{me^2}} = -\frac{1}{n^2}\cdot\underbrace{\frac{me^4}{2\hbar^2(4\pi\varepsilon_0)^2}}_{E_0 = 13.6\,eV}$$

$$E_n = -\frac{E_0}{n^2}$$

$$E_n = -\frac{1}{n^2}\cdot\frac{me^4 Z^2}{2\hbar^2(4\pi\varepsilon_0)^2} = -\frac{Z^2}{n^2}\cdot E_0$$

1.1.1 Predictions

Consider a scenario where a single electron orbits a nucleus containing Z protons.

- **Orbital Radius**

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{me^2 Z}n^2 = \frac{n^2}{Z}a_0,$$

where the Bohr radius of Hydrogen atom (with atomic number $Z = 1$) for $n = 1$ is $a_0 = 0.529\,\text{Å} = 0.529 \times 10^{-10}\ m$.

- **Energy Levels**

$$E_n = -\frac{me^4 Z^2}{2\hbar^2 n^2 (4\pi\epsilon_0)^2} = -\frac{Z^2}{n^2}E_0,$$

where the ionization energy of the Hydrogen atom is $E_n = 13.6\,\text{eV}$.

- **Speed of orbital electron**

$$v_n = \frac{e^2 Z}{4\pi\epsilon_0\hbar}\frac{1}{n} = \frac{Z}{n}\alpha c \equiv \frac{Z}{n}v_0,$$

where the fine structure constant is

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137}.$$

Note that the orbital angular momentum $L_n = mr_n v_n = n\hbar$, so that for $n = 1$ and $Z = 1$,

$$v_0 = \alpha c, \qquad a_0 = \frac{\hbar}{mv_0} = \frac{\hbar}{m\alpha c}, \qquad E_0 = \frac{1}{2}m(\alpha c)^2.$$

- An example of Lithium iron

Lithium (^{3}Li) has an atomic number of 3. Hence, its nucleus has 3 protons. Lithium-7 has an atomic mass of 7. Given that its atomic number is 3, there are 7 - 3 = 4 neutrons in its nucleus. Since atoms are charge neutral, the ^{3}Li atom has 3 electrons. Thus, the Lithium (^{3}Li^{++}) iron has one electron.

1.1.2 Fine structure constant

The fine-structure constant (α) is a dimensionless fundamental constant that characterizes the strength of electromagnetic interactions. Its expression in terms of Planck's constant (h) and the speed of light (c) is:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{\hbar c},$$

where:
- e is the elementary charge,
- ε_0 is the vacuum permittivity,
- $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant.
- Its current Value:

The latest CODATA (2018) recommended value of α is:

$$\alpha \approx 0.0072973525693 \quad (\text{approximately } \frac{1}{137.035999206}).$$

Running coupling constant