Universality and Emergence from On-Shell Recursion

Callum R. T. Jones

*Leinweber Center for Theoretical Physics*  
*Department of Physics, University of Michigan*

Based on work with: Henriette Elvang, Marios Hadjiantonis & Shruti Paranjape

hep-th: 1802.xxxxx, 1712.09937
A Proposal

- The way we think about, and work with, effective field theories for spontaneously broken symmetries is often unnecessarily complicated and obscures the essential physics.
- At very low energies the dynamics of Goldstone modes is universal and determined by the pattern of symmetry breaking:

\[
S = \sum_{[\mathcal{O}_i] \leq \Delta} \int d^4x \ g_i \mathcal{O}_i(x) + \sum_{[\mathcal{O}_i] > \Delta} \int d^4x \ g_i \mathcal{O}_i(x).
\]

- The universal part of the effective action generically has *miraculous* properties such as symmetries and conservation laws which only emerge at very low-energies.
- The universal part of the S-matrix is calculable entirely on-shell *without any explicit knowledge of the structure of the effective action* using modern recursive on-shell methods.
- From this point of view the emergence of low-energy symmetries is simple and manifest.
§ 1. Effective Field Theory on the Mass Shell
Some Historical Context

- The modern way of thinking about the dynamics of Goldstone modes is a product of a period of great anxiety from the 1950’s and 60’s arising from the apparent failure of local quantum field theory to describe Hadronic interactions.
- In the absence if a field theory description radical new approaches to particle physics like the original S-matrix theory of Chew and Mandelstam began to grow in popularity.
- A way forward was eventually found when it was realized that the lightest mesons were pseudo-Goldstone modes of a spontaneously broken $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$ chiral symmetry with the structure of low-energy scattering fixed by the algebraic structure of the commutators of the broken currents.
Low-Energy Theorems

• A pattern of symmetry breaking $G \rightarrow H$ (partially) fixes the algebra of Noether currents. Such a current algebra uniquely determines the universal low-energy dynamics:

$$[J^0_a(t, \vec{x}), J^0_b(t, \vec{y})] = i f_{abc} J^0_c(x) \delta^{(3)}(\vec{x} - \vec{y})$$

• **Spectrum**: Low-energy degrees of freedom are massless Goldstone modes

$$\langle 0 | J^\mu_a(x) | \pi_b(p) \rangle = i F \delta_{ab} p^\mu e^{i p \cdot x}.$$ 

• **Dynamics**: Low-energy (soft) theorems follow as consequences of Ward identities

$$\partial_\mu \langle J^\mu_b(x) J^\mu_{a1}(x_1)...J^\mu_{an}(x_n) \rangle = i \sum_{i=1}^{n} \delta^{(4)}(x - x_i) \langle J^\mu_{a1}(x_1)...\delta_a J^\mu_{ai}(x_i)...J^\mu_{an}(x_n) \rangle$$

If the coset $G/H$ is symmetric:

$$[T_{\text{broken}}, T_{\text{broken}}] \sim T_{\text{unbroken}} \quad (1)$$

then the soft limits of the corresponding amplitudes vanish (Adler zero):

$$A_n(p_b, ...) \rightarrow 0, \quad \text{as} \quad p \rightarrow 0.$$
How to think about EFTs On-Shell

• Direct use of current algebra is impractical.

• **Non-linear Realization:** Spontaneously broken currents should follow from *some* (non-linear symmetry) of *some* effective action

\[
S_{\text{eff}} \sim \int d^4x \left[ (\partial \pi)^2 + \frac{1}{F^2} \pi^2 (\partial \pi)^2 + \ldots \right], \quad \pi(x) \to F_{\pi} + \ldots
\]

<table>
<thead>
<tr>
<th>Off-shell</th>
<th>On-shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>S-matrix</td>
</tr>
<tr>
<td>Quantized fields</td>
<td>Asymptotic particles</td>
</tr>
<tr>
<td>Feynman diagrams</td>
<td>On-shell recursion</td>
</tr>
<tr>
<td>Symmetry (unbroken)</td>
<td>On-shell Ward identities</td>
</tr>
<tr>
<td>Symmetry (spontaneously broken)</td>
<td>Goldstone modes + low-energy theorems</td>
</tr>
</tbody>
</table>

• **New version of an old idea:** Invert our way of understanding spontaneous symmetry breaking. Goldstone modes and low-energy theorems *define* the S-matrix.

• Statements about the structure of the vacuum, current algebra and non-linearly realized symmetry are taken to be a secondary layer of interpretive structure.
§ 2. On-shell Recursion
Properties of the S-matrix

- **Analyticity**: Amplitudes are meromorphic functions of external momenta
  \[ p_i \rightarrow p_i + zq_i, \quad z \in \mathbb{C}, \quad A_n(p_1, \ldots, p_n) \rightarrow \hat{A}_n(z) \]

- **Locality**: Amplitudes have singularities only when on-shell states are produced
  \[ \hat{A}_n(z) = \sum_{I} \frac{c_I}{z - z_I} + \mathcal{O}(z^0), \quad \hat{P}_I^2(z_I) = 0 \]

- **Unitarity**: Amplitudes factorize into products on singularities
  \[ c_I = \sum_{|\psi^{(I)}\rangle} \frac{A_L^{(I)} A_R^{(I)}}{P_I^2} = \]

![Diagram showing factorization of amplitudes on singularities]
On-Shell Recursion

- Realize the amplitude as a contour integral

\[ A_n = -\sum_{i \in I} \text{Res}_{z=z_i} \left[ \frac{\hat{A}_n(z)}{z} \right] + \lim_{R \to \infty} \int_{C_R} \frac{dz}{2\pi i} \frac{\hat{A}_n(z)}{z} \]

\[ \sum_{I} \frac{A_L^{(I)} A_R^{(I)}}{P_f^2} \]

- **Constructibility Theorem:** Residue at infinity vanishes if

\[ 4 - n - [g] \left( \frac{n-2}{v-2} \right) - \sum_{i=1}^{n} s_i < 0. \]
Universality from Recursion

- **Constructibility Theorem**: S-matrix is recursively generated if

\[
4 - n - [g] \left( \frac{n - 2}{v - 2} \right) - \sum_{i=1}^{n} s_i < 0.
\]

- \( n \) = number of particles
- \( g \) = fundamental coupling
- \( v \) = valence of fundamental interaction
- \( s_i \) = spin of \( i \)th particle

- **Effective action**:

\[
S = \sum_{[O_i] \leq \Delta} \int d^4x \ g_i O_i(x) + \sum_{[O_i] > \Delta} \int d^4x \ g_i O_i(x) .
\]

- **Origin of Universality**: Lowest dimension operators dominate IR \( \Leftrightarrow \) S-matrix at leading order is recursive.
Emergence from Recursion

• **Constructibility Theorem:** S-matrix is recursively generated if

\[ 4 - n - [g] \left( \frac{n - 2}{v - 2} \right) - \sum_{i=1}^{n} s_i < 0. \]

• Only non-vanishing amplitudes are constructible by *gluing together* fundamental amplitudes.

\[
\begin{array}{c}
\mathcal{A}_L \rightarrow X \quad \overline{X} \quad \mathcal{A}_R \\
\vdots \quad \vdots
\end{array}
\]

States on each side of a factorization channel are CP conjugate.

• **Origin of Emergent Symmetry:** If the fundamental amplitudes conserve an additive quantum number and the S-matrix is recursive then that quantum number is conserved in all *tree-level* amplitudes.
**Example: Einstein-Maxwell**

- **In our universe** at very low energies the only on-shell degrees of freedom are the photon and graviton:

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{M_p^2}{8\pi} R + F^2 + g_4 F^4 + \ldots \right]
\]

- Fundamental interactions:

- Fundamental coupling \( A_3 \sim \frac{1}{M_p} \Rightarrow [g] = -1 \)

- Constructibility criterion:

\[
4 - n - (-1) \left( \frac{n - 2}{(3) - 2} \right) - n_\gamma - 2n_h = 2 - n_\gamma - 2n_h < 0.
\]

- **Emergence**: The optical helicity \( n_\gamma^+ - n_\gamma^- \) is conserved in the fundamental interactions. It is therefore conserved in all tree-level amplitudes. At very low-energies the total optical helicity is conserved in light-by-light scattering.
Subtracted Recursion

- First used with scalar EFTs [Cheung, Trnka et al. 1611.03137, 1509.03309]
- **Key Idea:** Assume low-energy theorem. Make a special deformation

\[ p_i \rightarrow (1 - a_i z)p_i, \quad \text{where} \quad \sum_{i=1}^{n} a_i p_i = 0. \]

- Realize the amplitude as a contour integral with *subtractions*

\[
\mathcal{A}_n = \sum_{i \in I} \Res_{z = z_i} \left[ \frac{\hat{A}_n(z)}{z(1 - a_1 z)^{\sigma_1}...(1 - a_n z)^{\sigma_n}} \right] + \lim_{R \to \infty} \oint_{C_R} \frac{dz}{2\pi i} \frac{\hat{A}_n(z)}{z(1 - a_1 z)^{\sigma_1}...(1 - a_n z)^{\sigma_n}}
\]

\[
\sum_{I} \frac{A_L^{(I)} A_R^{(I)}}{P^2_I}
\]

- No additional residues at \( z = 1/a_i \), amplitude has zero of order \( \sigma_i \).
- **Constructibility Theorem (Generalized):** Residue at infinity vanishes if

\[
4 - n - [g] \left( \frac{n - 2}{v - 2} \right) - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0.
\]
EFT Bootstrap

- **Key idea:** Not all choices of initial conditions (fundamental amplitudes) are consistent.

\[
\mathcal{A}_n = \sum_{i \in I} \text{Res}_{z=z_i} \left[ \frac{\hat{A}_n(z)}{z(1-a_1 z)^{\sigma_1} \cdots (1-a_n z)^{\sigma_n}} \right], \quad \sum_{i=1}^{n} a_i p_i = 0.
\]

- Left-hand-side *manifestly* independent of \( a_i \).

- Numerical bootstrap:
  - Assume a spectrum of particles and low-energy theorems
  - Make the most general ansatz for the fundamental interactions
  - Numerically calculate the right-hand-side and verify \( a_i \)-independence

![Graph](image-url)
§ 3. Applications
Chiral Perturbation Theory

- Symmetry breaking pattern: $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$.
- Spectrum: Goldstone bosons $\pi^a$ in the adjoint representation of $SU(N_f)_V$.
- Low energy theorem (Adler zero):

$$A_n (\ldots \{\epsilon p_i\} \pi \ldots ) \sim \epsilon^1, \quad \epsilon \sim 0 \quad \Rightarrow \quad \sigma_\pi = 1.$$

- Fundamental interactions:

$$A_4 (\pi \pi \pi \pi) \sim 1/F^2_\pi$$

- Constructibility criterion:

$$4 - n - (-2) \left( \frac{n - 2}{(4) - 2} \right) - \sum_{i=1}^{n} (0) - \sum_{i=1}^{n} (1) = 2 - n < 0.$$
Chiral Perturbation Theory

\[ S = F_\pi^2 \int d^4 x Tr \left[ \partial_\mu U^\dagger \partial^\mu U \right] + \frac{2N_c}{15\pi^2 F_\pi^5} \int d^4 x \epsilon^{\mu\nu\rho\sigma} Tr \left[ \pi \partial_\mu \pi \partial_\nu \pi \partial_\rho \pi \partial_\sigma \pi \right] + \ldots \]

\[ \Longleftrightarrow \quad A_n \sim \frac{1}{F_\pi^{n-2}} A_n^{(0)} + \frac{1}{F_\pi^n} A_n^{(1)} + \ldots \]

- Recursion from 4-point amplitudes

- **Emergence:** Only even amplitudes can be constructed \( \Rightarrow \) at very low energies \((-1)^n\) is conserved.
Partially Broken Supersymmetry

- Symmetry breaking: $\mathcal{N} = 2$ supersymmetry spontaneously broken to $\mathcal{N} = 1$.
- Spectrum:

\[
\mathcal{N} = 2 \rightarrow \mathcal{N} = 1
\]

Vector Goldstone multiplet

\[
\psi + \gamma
\]

Chiral Goldstone multiplet

\[
\phi + \psi
\]

- Low energy theorem:

\[
\mathcal{A}_n \left( \ldots \{e^|i\rangle, |i\rangle \}^+_{\psi} \ldots \right) \sim \epsilon^1, \quad \epsilon \sim 0 \quad \Rightarrow \quad \sigma_{\psi} = 1.
\]

- Fundamental interactions (compatible with $\mathcal{N} = 1$ supersymmetry):

\[
\begin{array}{cccc}
\gamma^+ & \gamma^+ & \gamma^+ & \gamma^+ \\
A_4 & A_4 & A_4 & A_4 \\
\gamma^- & \gamma^- & \psi^- & \psi^- \\
\end{array}
\]
Partially Broken Supersymmetry

- Fundamental coupling $A_4 \sim \frac{1}{\Lambda^4} \Rightarrow [g] = -4$.
- Constructibility criterion:

$$4 - n - (-4) \left( \frac{n - 2}{4 - 2} \right) - \frac{1}{2} n_{\psi} - n_\gamma - (1)n_{\psi} = -\frac{1}{2} n_{\psi} < 0.$$

- Supersymmetry fixes $n_{\psi} = 0$ amplitudes in terms of the others.
Partially Broken Supersymmetry

- Fundamental interactions separately conserve:
  - Optical helicity: \( n_+ - n_- \)
  - Chiral symmetry: \( n_+ - n_- \)

- **Emergence**: Any model with \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \) spontaneous supersymmetry breaking must have both an emergent *chiral symmetry* and an emergent *electric-magnetic duality* at very low energies.
Internal symmetry breaking with $\mathcal{N} = 2$ SUSY

- Symmetry breaking: $SU(2) \rightarrow U(1)$ internal symmetry breaking, with unbroken $\mathcal{N} = 2$ supersymmetry.
- Spectrum: Complex scalar Goldstone bosons $\phi \subset \mathcal{N} = 2$ vector multiplet $\phi + \psi_{1,2} + \gamma$.
- Low energy theorem:

$$A_n \left( \ldots \{ \epsilon p_i \} \phi \ldots \right) \sim \epsilon^1, \quad \epsilon \sim 0 \quad \Rightarrow \quad \sigma_{\phi} = 1.$$  

- Fundamental interactions (compatible with $\mathcal{N} = 2$ supersymmetry):

\[
\begin{align*}
A_3 & \rightarrow \phi \\
A_3 & \rightarrow \bar{\phi} \\
A_3 & \rightarrow \gamma^+ \\
A_3 & \rightarrow \gamma^- \end{align*}
\]
Internal symmetry breaking with $\mathcal{N} = 2$ SUSY

- Fundamental coupling: $\mathcal{A}_3 \sim \frac{1}{\Lambda} \Rightarrow [g] = -1$.
- Constructibility criterion:

$$4 - n - (-1) \left( \frac{n - 2}{3} - 2 \right) - \frac{1}{2} n_\psi - n_\gamma - (1) n_\phi = 2 - \frac{1}{2} n_\psi - n_\gamma - n_\phi < 0.$$  

- Supersymmetry fixes non-constructible amplitudes in terms of constructible ones.
Internal symmetry breaking with $\mathcal{N} = 2$ SUSY

• Fundamental interactions conserve the following charges:

\[
q[\phi] = +4, \quad q[\psi_+^A] = +1, \quad q[\gamma^+] = -2 \\
q[\bar{\phi}] = -4, \quad q[\psi_-^A] = -1, \quad q[\gamma^-] = +2
\]

• **Emergence:** Any $\mathcal{N} = 2$ supersymmetric model which realizes the spontaneous global symmetry breaking pattern $SU(2) \rightarrow U(1)$ must have an emergent electric-magnetic duality at very low-energies with the given charge assignments.
§ 4. Conclusions and Lessons
Conclusions and Lessons

• There is a great deal to be learned by reconceptualizing the physics of spontaneous symmetry breaking on-shell.

• Low-energy theorems directly control the universal structure of the effective action at very low-energies.

• This universality is encoded in S-matrix as on-shell recursion.

• Moreover the recursive structure reveals hidden symmetries which are emergent at very low-energies.

• We have:
  - Derived a precise theorem from the principles of S-matrix theory for the validity of on-shell recursion in massless theories (both regular and subtracted).
  - Extended the method of subtracted recursion to models with massless particles of all spins.
  - Applied the numerical EFT bootstrap to fermionic models (work in progress).
  - Given an explicit on-shell construction of the S-matrix in models of partially spontaneously broken supersymmetry.
  - Uncovered many examples of emergent electric-magnetic duality symmetries and given a novel method for future discovery.
Relation to Other Work

• 1712.09937 *On the Supersymmetrization of Galileon Theories in Four Dimensions*
  Henriette Elvang, CRTJ, Marios Hadjiantonis & Shruti Paranjape
  
  ○ Galileons are scalar EFTs with higher-derivative interactions (but only second order equations of motion) used in cosmological model building.
  ○ We applied on-shell methods (primarily the numerical EFT bootstrap) to constrain the possible form of $\mathcal{N} = 1$ supersymmetrizations of Galileon models.

• 1611.07534 *Soft Photon and Graviton Theorems in Effective Field Theory*
  Henriette Elvang, CRTJ & Stephen G. Naculich
  
  ○ Photons and gravitons satisfy universal (singular) low-energy theorems.
  ○ Recent revival in interest due to connections to spontaneously broken asymptotic symmetries.
  ○ We showed that in a general EFT the soft photon and graviton theorems are modified at sub- and sub-sub-leading order respectively by non-universal contributions from a family of higher-dimension operators.
  ○ Discovered a surprisingly rich set of constraints on local, unitary quantum field theory including a novel proof of the Weinberg-Witten theorem.
Thank you!