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LETTER TO THE EDITOR

Finite size scaling approach to Anderson localisation

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Abstract. Using a new and powerful real space renormalisation method recently introduced for phase transitions, we prove numerically that two dimensions are marginal for Anderson localisation. An associated transition is found, presenting close analogies with the Kosterlitz–Thouless transition of the XY model in two dimensions, while the behaviour in three dimensions is shown to be a standard one with a mobility edge.

The accuracy of the method is better than previous approaches in two dimensions and is not reduced when the number of dimensions is raised to three.

The purpose of this Letter is to draw attention to an approach to Anderson localisation based on the possibility of obtaining the critical properties of higher-dimensional systems from one-dimensional models and on the study of Lyapunov characteristic exponents (LCE) relative to the asymptotic behaviour of random matrix products.

The first concept used in this new approach is to extract the nature of the states of the infinite lattice by finite size scaling from the variation of characteristic lengths with lattice size. This 'phenomenological renormalisation theory' has been illustrated by application to the Ising model (Nightingale 1976), to the percolation problem (Derrida and Vannimenus 1980), to the O(N) Heisenberg Hamiltonians (Hamer and Barber 1980) and to self-avoiding walks in two dimensions (Derrida 1981).

Let us explain it in the case of the exactly solvable Ising model on a square lattice (Nightingale 1976). It consists of studying the behaviour of the correlation length $\xi_n(T)$ along a strip of width n against n for different temperatures T (figure 1).

For $T > T_c$, ξ_n converges towards the correlation length of the infinite square lattice and sufficiently broad strips simulate the infinite lattice. $\xi_n(T)$ grows linearly with n at the critical temperature T_c and an exponential increase is obtained for $T < T_c$. At those temperatures where $\xi_n(T)$ is much larger than n, the infinite lattice cannot be simulated by a strip, but a scaling theory can be developed (Fisher and Barber 1972).

Let us recall the salient features of this theory. When $T > T_c$, the correlation length $\xi_n(T)$ of a strip of width n can be described by the *ansatz:*

$$\xi_n(T) = \xi_\infty(T) f(n/\xi_\infty(T)) \qquad (n \to \infty,\ T \to T_c^+)$$

where $f(x)$ is an unknown function of the dimensionless ratio n/ξ_∞ and ξ_∞ is the correlation length of the infinite lattice. It is legitimate to assume for f two limiting behaviours: when x goes to infinity, $f(x)$ goes to unity, and when x goes to zero, $f(x)$ behaves as x^α. When T decreases to T_c, ξ_∞ diverges as $(T - T_c)^{-\nu}$, while $\xi_n(T)$ and n remain finite, so that we must take α equal to unity. In other words, the critical temperature appears to be

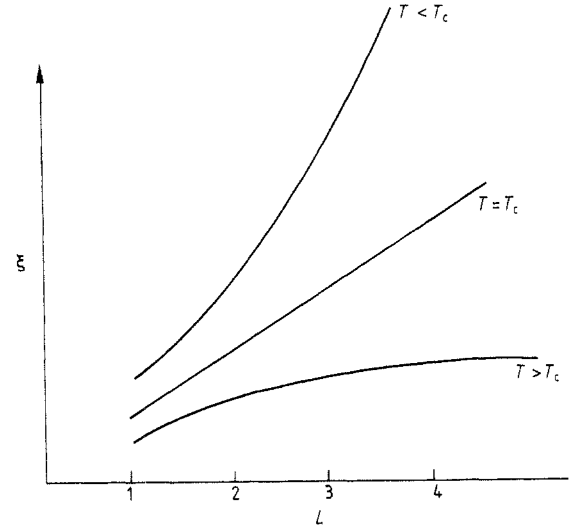

Figure 1. Characteristic behaviour of the correlation length of a strip against the width l in a standard phase transition (e.g. Ising model in 2D).

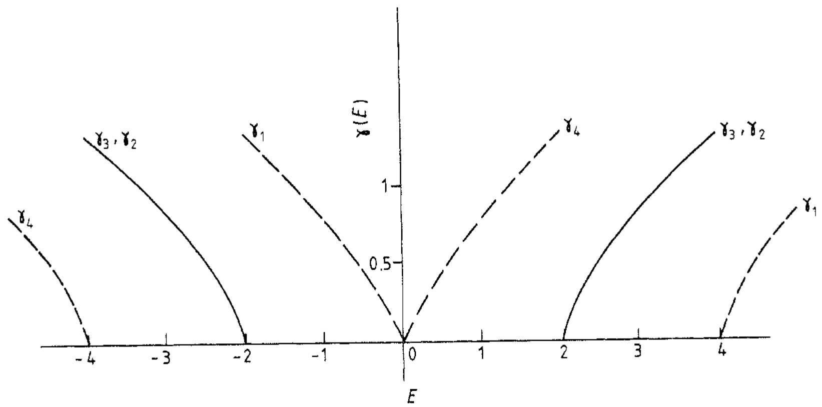

Figure 2. Lyapunov exponents γ of a strip of width 4 against the energy in the case without disorder ($W = 0$). All the states are plane waves and only the smallest characteristic exponent $\gamma_{\min}(E)$ possesses physical meaning.

the fixed point of the transformation:

$$T' = R_{n,m}(T)$$

obtained from the relation

$$\xi_n(T)/\xi_m(T') = n/m.$$

In all the above mentioned applications of this method, convergence is obtained quickly against n and fairly accurate approximations of the fixed point and related exponents are found even for small values of n.

For Anderson localisation, we have to define the characteristic length corresponding to the correlation length and the inverse of the smallest Lyapunov characteristic exponent is the second concept that we need. On the one hand, Furstemberg's theorem applied to transfer matrix products proves localisation in one-dimensional random models (Ishii 1973). On the other hand, the only exactly solvable (but non-random) model that we know at the present time and which exhibits a transition from extended to exponentially localised wave functions (Aubry and André 1980) illustrates that the concept of LCE is meaningful and efficient in the study of localisation.

For definiteness, let us consider on an infinite strip of width l the Anderson model represented in the tight-binding approximation by the stationary Schrödinger equation:

$$E\psi_{n,m} = \varepsilon_{n,m}\psi_{n,m} + \psi_{n,m+1} + \psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m-1}$$

where $\varepsilon_{n,m}$ is distributed randomly between $-W/2$ and $W/2$. The lateral boundary conditions are taken to be periodic ($\psi_{n,m+l} = \psi_{n,m}$). When n goes to infinity, we shall investigate the asymptotic behaviour of the vector $\Lambda_n(E)$, whose $2l$ components are:

$$(\psi_{n+1,1};\ldots;\psi_{n+1,l};\psi_{n,1};\ldots;\psi_{n,l})$$

obtained from a particular vector $\Lambda_0(E)$ by the matrix product $M_n = \prod_{i=1}^{n} T_i$, where T_i is a real $(2l \times 2l)$ transfer matrix given by

$$T_i = \begin{pmatrix} P_i & -I \\ I & 0 \end{pmatrix} \text{ and } P_i = \begin{pmatrix} (E-\varepsilon_{i,1}) & -1 & 0\ldots\ldots 0 & -1 \\ -1 & (E-\varepsilon_{i,2}) & & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & & & -1 \\ -1 & 0\ldots\ldots\ldots\ldots 0 & -1 & (E-\varepsilon_{i,l}) \end{pmatrix}.$$

These transformations and their product are symplectic, i.e. their eigenvalues are pairs whose elements are the inverse of each other and it is sufficient to know the first lth eigenvalues.

Oseledec's theorem states (Oseledec 1968) that an asymptotic matrix Γ exists, defined by

$$\Gamma = \lim_{n\to\infty} (M_n^* M_n)^{1/2n}$$

where the asterisk denotes a matrix transposition.

If we call $\exp\gamma_1 \ldots \exp\gamma_{2l}$ the eigenvalues of Γ and $v_1 \ldots v_{2l}$ the corresponding eigenspaces, then the property

$$\lim_{n\to\infty} (1/n) \ln\|M_n u\| = \gamma_r$$

holds for any vector $u \in v_r$. These γ_r are the LCE of the random matrix product M_n. This implies, with full mathematical rigor, for any system as a strip or a bar, that any arbitrary vector $\Lambda_0(E)$ is a linear combination of a set of vectors whose asymptotic divergence is given by the different LCE γ_r. This process is called a filtration in dynamical systems (Benettin and Galgani 1979).

Oseledec's theorem gives the rigorous answer to the recently widely discussed problem (Anderson *et al* 1980) of finding the correct 'scaling' variables (or those which possess the property of 'additive mean'). This same theorem also shows why it is meaningless to average $M_n^* M_n$ (Abrahams and Stephen 1980, Andereck and Abrahams 1980, Stephen 1980). Indeed such an average, which can be done easily, violates the fundamental property that those matrix products are symplectic.

However, we insist upon the fact that in our calculations no use is made of any questionable averaging procedure, since they rely on the existence of Oseledec's asymptotic matrix Γ.

In this Letter we focus our interest on L_{max}, defined as the inverse of the smallest LCE, which represents the largest possible localisation length for any vector $\Lambda_0(E)$, and consequently for an eigenvector, if it exists at that given energy.

By analogy with the above mentioned applications of the finite size scaling theory, we can expect three kinds of behaviours when we study L_{max} against the width l of a strip for localisation on a square lattice, or against the side s of a bar for localisation in a cubic lattice.

If L_{max} against increasing l or s converges towards a finite limit, it proves that any eigenstate, if it exists at that given energy, converges exponentially to zero at infinity with a localisation length at most equal to that finite limit (behaviour I).

If L_{max} against increasing l or s diverges, we have to distinguish between two kinds of possible divergences:

(i) A linear one, which means scale invariance as in the Ising model at the critical temperature. Consequently, the possible eigenstate corresponding to L_{max} is not exponentially localised (because the extrapolated localisation length is infinite), nor is it

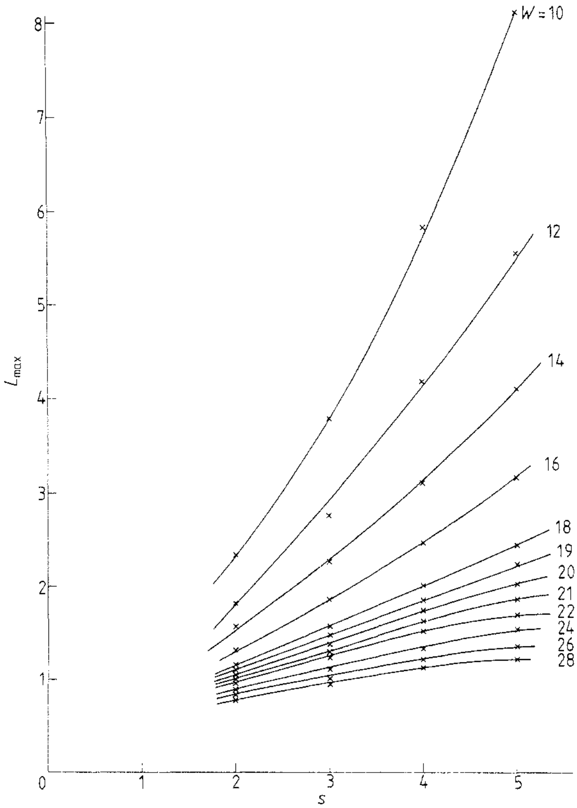

Figure 3. L_{max} against the side s of a bar for an energy equal to zero and periodic lateral boundary conditions.

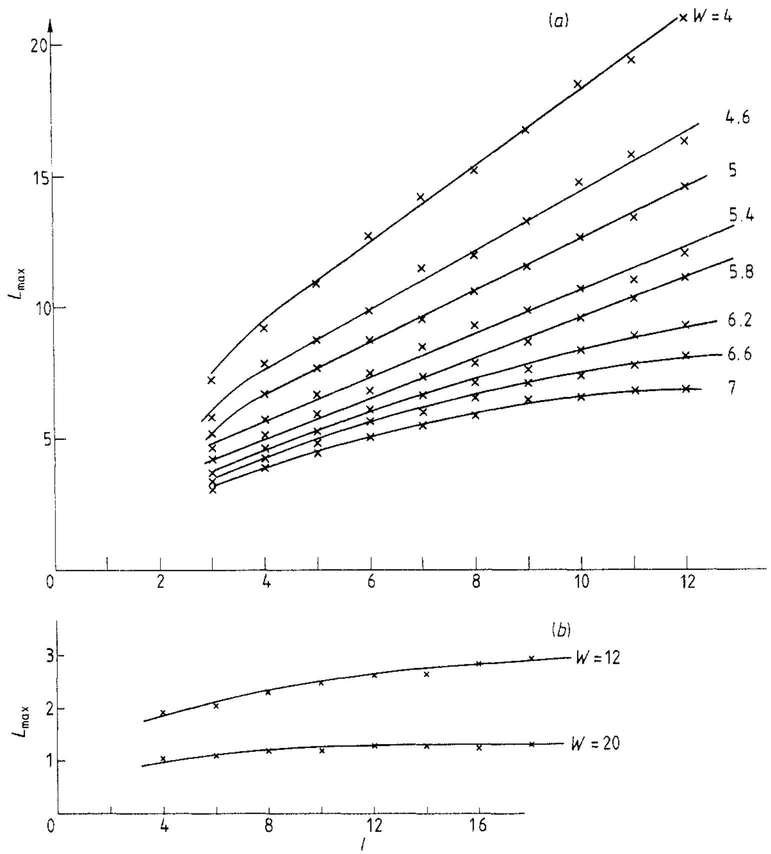

Figure 4. (*a*) $L_{\max}$ against the width l of the strip for an energy equal to zero; (*b*) behaviour I in two dimensions for $E = 0$.

extended. The linear growth of $L_{\max}$ corresponds to a singular state which decays to zero at infinity with a power law, in the same way as the correlation function of the Ising model at T_c (behaviour II).

(ii) A faster than linear divergence of $L_{\max}$ proves that there exist at that given energy at least two extended states. Indeed, in that case, the smallest LCE being zero, it is possible to build up two different extented states which correspond to the two possible ways of transfer along the one-dimensional system. Those states do not decay to zero at infinity (behaviour III).

The LCE have been calculated with the help of a method described in the case of an ergodic problem of Hamiltonian and dissipative dynamical systems (Benettin and Galgani 1979, Shimada and Nagashima 1979)†, and a quite satisfactory convergence after a product of 10^4 matrices was found for distinguishing between the three announced kinds of behaviour.

In order to compare our results with those of Lee (1979) in two dimensions, we have fixed the energy E equal to zero and taken W as a variable.

† We emphasise that our numerical process is based on a rigorous mathematical method which deduces the first s LCE from the asymptotic divergence of a volume defined by s orthogonal vectors.

By the study of random transfer matrix products on bars (figure 3), we find that in the case of three dimensions, there is only one critical value, $W_c \sim 19 \pm 1$, for which the linear growth is observed, separating a region of exponentially localised states ($W > W_c$, behaviour I) from a region of extended states ($W < W_c$, behaviour III).

On the other hand, our studies on strips (figure 4) give a very different result for two dimensions. We have found a critical value of W ($W_c \sim 6 \pm 0.2$) in close agreement with the value given by Lee (1979) separating two regions.

For $W > W_c$, behaviour I is found. For *all* $W < W_c$, behaviour II is found. This result, which is very similar to the one found by Hamer and Barber in the O(2) Heisenberg Hamiltonian in $(1 + 1)$ dimensions, must be understood as a spectacular demonstration of the existence of a whole region of scale invariance in two dimensions, and is by no means in contradiction with the scaling theory result (Abrahams *et al* 1979).

Actually, the natural conclusion for two dimensions is that the states are exponentially localised for $W > W_c$, and that for all $W < W_c$ the states are 'singular', decaying to zero with a power law. This situation is reminiscent of the 'quasi-order' transition of the 2D XY model (Kosterlitz and Thouless 1973).

Our results in two dimensions thus confirm the recent ideas postulated by Abrahams, Anderson, Licciardello and Ramakrishnan on the nature of the eigenfunctions, and also give a solution to the controversy between Lee (1979) and Abrahams *et al* (1979).

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