

PHY481 - Lecture 7: The electrostatic potential and potential energy

Griffiths: Chapter 2

B. Electric potential energy and electric potential

Physical definition

The electric potential energy (U) is the potential energy due to the electrostatic force. As always only differences in potential energy correspond to physical observables. However we define a reference potential energy and calculate all differences in potential energy with respect to this reference. In electrostatics, the potential energy is usually, but not always (see below), defined to be zero when the charges are an infinite distance apart.

The difference in potential energy in moving a charge between two positions a and b is defined in terms of the work done in moving the charge between these two positions, so that,

$$\Delta U_{ab} = \int_a^b \vec{F}_{ext} \cdot d\vec{l} = - \int_a^b \vec{F} \cdot d\vec{l} \quad (1)$$

where \vec{F} is the Coulomb force on the charge and $d\vec{l}$ is a small element of distance along the path we choose to take from a to b . Note that Eq. (1) does not depend on the path we take. All paths from a to b give the same change in potential energy. Note also that Eq. (1) is true only for conservative forces, ie. systems in which there is no dissipation (e.g. friction).

Instead of using the Coulomb force, we defined the electric field which is the force per unit charge. In the same way we define the electric potential, V , to be the potential energy per unit charge, ie. $U = qV$. The electric potential is so important it is given its own unit, the volt (V). In terms of the electric potential, equation (1) is,

$$\Delta V_{ab} = - \int_a^b \vec{E} \cdot d\vec{l} \quad (2)$$

From this equation we see that the units of voltage are Nm/C . We have defined the volt to be the unit of voltage, so we have the relation $V = Nm/C$ or $V/m = N/C$. So far we have used N/C as the units of electric field. This is usual in electrostatics. However in circuit theory it is usual to use volts/meter (V/m).

If we move a charge q from a point a to a point b , then the change in potential energy, U_{ab} , of the system is

$$\Delta U_{ab} = - \int_a^b \vec{F} \cdot d\vec{l} \quad (3)$$

- \vec{F} is the electric force on charge q due to all of the other charges in the system

- $d\vec{l}$ is a small displacement along the path we choose to take from a to b .

If we define the electric potential to be the potential energy per unit charge, so that $V = U/q$, then the difference in electric potential is given by,

$$\Delta V_{ab} = \frac{1}{q} \Delta U_{ab} = - \int_a^b \vec{E} \cdot d\vec{l} \quad (4)$$

Often, we define the potential energy and hence the potential of a charge at infinity to be zero, however in some cases this is not possible for mathematical reasons.

With the definition of potential given in Eq. (4) we can calculate the potential difference between any pair of positions.

Mathematical Derivation

The electrostatic potential can also be deduced on purely mathematical grounds using the relation $\vec{\nabla} \wedge \vec{E} = 0$. This equation is satisfied when $\vec{E} = -\vec{\nabla}V$ due to the vector identity $\vec{\nabla} \wedge (\vec{\nabla}V) = 0$ which holds for any scalar function V . In terms of the electrostatic potential. Integrating $\vec{E} = -\vec{\nabla}V$ leads to Eq. (4) (see Eq. (1.55) of Griffiths). Note that using $\vec{E} = -\vec{\nabla}V$, the differential form of Gauss's law then becomes,

$$\nabla^2 V = -\rho/\epsilon_0. \quad (5)$$

This is called Poisson's equation and is the most commonly solved form of Gauss's law. The special case $\rho = 0$ is also very important and is called Laplace's equation. In fact most of Chapter 3 of Griffiths is devoted to Laplace's equation. Laplace's equation is most non-trivial when boundary conditions corresponding to finite charge or

electrostatic potential are used. We then need to solve boundary value problems of partial differential equations.

Methods for calculating the electrostatic potential

1. *Direct method:* Find the electric field by using e.g. Gauss's law and then carry out the integral,

$$V(r) = - \int_{\infty}^r \vec{E} \cdot d\vec{l} \quad (6)$$

If this integral diverges, we cannot use a reference potential at infinity, but instead can only talk of voltages across finite distances. For example if the electric field is a constant and directed in the \hat{r} direction, then we have,

$$V_{ab} = - \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} \rightarrow -E(r_b - r_a) = -Ed \quad \text{if } E \text{ is constant} \quad (7)$$

where $d = r_b - r_a$.

2. *Superposition:* Do a scalar addition of the contributions from each charge,

$$V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|} \rightarrow \int d\vec{r}' \frac{k\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (8)$$

where the last expression applies to a continuous charge distribution with charge density ρ . In Chapter 3 we will carry out a multipole expansion based on superposition.

3. *Differential form:* Here we use Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (9)$$

Note that since we know that for a point charge, Q , at position \vec{r} , $\rho(\vec{r}) = Q\delta(\vec{r} - \vec{r}')$, then the solution to this equation for this case is $V(|\vec{r} - \vec{r}'|) = kQ/|\vec{r} - \vec{r}'|$.

4. *Image charge method:* This is a special method that is very useful in many problems with special symmetry. It will be discussed in detail in Chapter 3.

Integrating the electric field to find the potential difference

(i) *Electric potential due to a point charge*

The electric field due to a point charge is

$$\vec{E} = \frac{kQ\hat{r}}{r^2} \quad (10)$$

The electric potential due to a point charge is then,

$$V = - \int_{\infty}^r \frac{kQ\hat{r}}{r'^2} \cdot d\vec{l} = - \int_{\infty}^r \frac{kQ\hat{r}}{r'^2} \cdot dr'\hat{r} = -kQ \int_{\infty}^r \frac{dr'}{r'^2} = \frac{kQ}{r} \quad (11)$$

(ii) *Electric potential inside and outside a uniform shell of charge*

Take the shell to have radius R and total charge Q . As for a point charge the integration is taken from ∞ to the point r . In the regime $r > R$ the integration is the same as that for a point charge, so we find that $V(r > R) = kQ/r$. For $r < R$, the part of the path in the regime $r < R$ makes no contribution as the electric field in this regime is zero. We therefore have $V(r < R) = kQ/R$. The electric potential for two concentric uniform shells, with radii R_1 and R_2 (with $R_1 > R_2$), and charges Q_1 and Q_2 . We can use superposition to find,

$$V(r < R_2) = kQ_2/R_2 + kQ_1/R_1; \quad V(R_2 < r < R_1) = kQ_1/R + kQ_2/r; \quad V(r > R_1) = k(Q_1 + Q_2)/r \quad (12)$$

(iii) *Electric potential for a uniform cylinder of charge*

The electric field is $\vec{E} = \lambda\hat{s}/(2\pi\epsilon_0 s)$. The potential difference between two points $b > s > s_1$ is then,

$$V_{b>s>s_1} = - \int_b^s \frac{\lambda}{2\pi\epsilon_0 s'} ds' = \frac{\lambda}{2\pi\epsilon_0} \ln(b/s) \quad (13)$$

The potential for $s < s_1$ is the same as that at $s = s_1$, ie. $\lambda \ln(b/s_1)/(2\pi\epsilon_0)$. Notice that in this problem we do not take infinity as the reference point as in that limit this expression diverges. This is not unphysical as we can never make a truly infinite line of charge. Instead this formula only applies in regimes where both b and s are much smaller than the length of the cylinder of charge. Using superposition we can write down the potential for the case of two concentric cylinders of radii s_1 and s_2 and charge per unit length λ_1 and λ_2 ,

$$V(s < s_2) = \frac{1}{2\pi\epsilon_0}[\lambda_1 \ln(b/s_1) + \lambda_2 \ln(b/s_2)];$$

$$V(s_2 < s < s_1) = \frac{1}{2\pi\epsilon_0}[\lambda_1 \ln(b/s_1) + \lambda_2 \ln(b/s)]; \quad V(s > s_1) = \frac{(\lambda_1 + \lambda_2)}{2\pi\epsilon_0} \ln(b/s) \quad (14)$$

(iv) *Electric potential for a uniform sheet of charge*

Consider an infinite, uniform sheet of charge with charge density σ lying in the x-y plane. The electric potential between two points $b > z$ on the positive z-axis is given by,

$$V(b - z) = - \int_b^z \frac{\sigma}{2\epsilon_0} \hat{z} \cdot dz \hat{z} = \frac{\sigma}{2\epsilon_0} (b - z). \quad (15)$$

This is an example of a very important case, name that of a constant electric field. In general if there is a constant electric field $\vec{E} = E_0 \hat{z}$, then the electrostatic potential is $-E_0 z + \text{constant}$. This form of the potential (usually with $\text{constant} = 0$) will be used extensively in Chapter 3.

Superposition method

More complex systems where the symmetry is not high enough to find the electric field using Gauss's law, it is often better to find the electrostatic potential first. The math is a little easier as the electrostatic potential is a scalar. Consider for example a uniform ring of charge, with charge per unit length (λ) and radius R . We calculated the electric field on the z axis in Lecture 5 (see Eq. (7), (8) of the lecture notes). To find the potential on the z-axis, we carry out a similar calculation,

$$V(z) = \int \frac{k dQ(\phi)}{|r|} = \int_0^{2\pi} k \frac{\lambda R d\phi}{r} = \frac{kQ}{(R^2 + z^2)^{1/2}} \quad (16)$$

where we used $Q = 2\pi R\lambda$. It is easy to check that the electric field $E_z = -\partial V/\partial z$ found from $V(z)$ is the same as Eq. (7) of Lecture 5. Another good example is a finite rod of charge. This can also be used to find the potential due to a square loop of charge.

We have not yet carried out superposition calculations for three dimensional charge distributions - for the electric field case we only considered one dimensional and two dimensional cases. Three dimensional cases are also often considered so it is good to see how it works. The simplest case is a uniform sphere of charge, which can be solved using Gauss's law and then integration, so we can check the result.

Lets consider a less trivial case, namely a uniform shell of charge density σ (Example 7 of Chapter 2 of Griffiths). Of course we know the answer from using Gauss's law, so we can check our result. The procedure we use here will also be used extensively in the image charge calculations of Chapter 3. If we are interested in a general point \vec{r} , without loss of generality, we can choose \vec{r} to lie on the z-axis. The superposition expression for the potential is then,

$$V(r) = \int k \frac{dQ(\theta', \phi') d\vec{r}'}{|\vec{r} - \vec{r}'|} = k\sigma \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin\theta' d\theta' d\phi'}{(R^2 + r^2 - 2Rr \cos(\theta'))^{1/2}} \quad (17)$$

The ϕ' integral gives 2π , while the θ' integral is also easy as $2Rr \sin(\theta')$ is the derivative of the argument of the denominator, so we find,

$$V(r) = 2\pi k\sigma \frac{R^2}{Rr} (R^2 + r^2 - 2Rr \cos(\theta'))^{1/2} \Big|_0^\pi$$

$$= 2\pi k\sigma \frac{R}{r} [(R^2 + r^2 + 2Rr)^{1/2} - (R^2 + r^2 - 2Rr)^{1/2}] = 2\pi k\sigma \frac{R}{r} [R + r - |R - r|] \quad (18)$$

For $r > R$, we take the negative sign in the modulus, so that $V(r > R) = 4\pi k\sigma R^2/r = kQ/r$. For $r < R$, we take the positive sign in the modulus, so that $V(r < R) = kQ/R$, as expected. Problem 2.28 in Homework 2 asks that you extend this calculation to the case of a uniform sphere of charge.