Solutions to problems for Part 2

Solutions to Quiz 3 are at the end of problems.

Sample Quiz Problems

Quiz Problem 1. Write down the equation for the thermal de Broglie wavelength. Explain its importance in the study of classical and quantum gases.

Solution

$$\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2} \tag{1}$$

This is of the form h/p_T, where $p_T = (2\pi m k_B T)^{1/2}$ is an average thermal momentum. Define the average interparticle spacing of a gas $L_c = (V/N)^{1/3}$. If $\lambda > L_c$ quantum effects become important in the thermodynamics.

Quiz Problem 2. Why are the factors $1/N!$ and $1/h^{3N}$ introduced into the derivation of the partition function of the ideal classical gas?

Solution

The factor $1/N!$ is needed to account for the fact that when an intergration is carried out over all phase space for N particles, all permutations of the particle identities is included. For indentical particles this must be removed. The factor $1/h^{3N}$ takes account of the Heisenberg uncertainty principle which states that the smallest phase space volume that makes sense is $(\hbar/2)^3$. The fact that it is $1/h^3$ instead of $1/(\hbar/2)^3$ for each particle is to reproduce the high temperature behavior of quantum gases.

Quiz Problem 3. By using the fact the $g_{3/2}(1) = \zeta(3/2) = 2.612$ and using,

$$N = \frac{V}{\lambda^3} g_{3/2}(z) + \frac{1}{1-z} = N_1 + N_0 \tag{2}$$

find an expression for the critical temperature of the ideal Bose gas in three dimensions.

Solution The condition for Bose condensation is $z = 1$ and

$$N_1 = N, \quad \text{or} \quad N = \frac{V}{\lambda_C^3} g_{3/2}(1) \tag{3}$$

Solving for T_C gives,

$$T_C = \frac{h^2}{2\pi m k_B}\left(\frac{N}{V\zeta(3/2)}\right)^{2/3} \tag{4}$$

Quiz Problem 4. State and give a physical explanation of the behavior of the chemical potential μ and the fugacity $z = e^{\beta\mu}$ as temperature $T \to \infty$, for both the Bose and Fermi gases.

Solution In the high temperature limit we can understand the behavior of μ by considering the grand potential,

$$\Phi_G = -PV; \quad \mu = \left(\frac{\partial \Phi_G}{\partial N}\right)_{T,V} = -V\left(\frac{\partial P}{\partial N}\right)_{T,V} \tag{5}$$

The derivative is positive at high temperatures as the pressure increases with the addition of particles, therefore μ is large and negative. The physical origin of this effect is that as particles are moved from a reservoir to the system a large reduction in total kinetic energy occurs at high temperature. This is true for both Bose and Fermi gases.

Quiz Problem 5. State and give a physical explanation of the behavior of the chemical potential μ and the fugacity $z = e^{\beta\mu}$ as temperature $T \to 0$, for both the Bose and Fermi gases.

Solution.
For the Bose gas as temperature goes to zero, the internal energy contribution dominates. As temperature goes to zero all of the particles that are added go into the ground state, so the chemical potential goes to the ground state energy. For the ideal gas case the ground state energy is zero, so the chemical potential goes to zero. The fugacity therefore goes to one.

For the Fermi case the lowest unoccupied state is at the Fermi energy so as particles are added to the system, the energy changes by ϵ_F. The Fermi energy is positive so $\beta\mu$ becomes large at low temperature and hence $z = e^{\beta\mu}$ increases very rapidly as $T \to 0$.

Quiz Problem 6. Write down the starting expression in the derivation of the grand partition function, Ξ_F for the ideal Fermi gas, for a general set of energy levels ϵ_l. Carry out the sums over the energy level occupancies, n_l and hence write down an expression for $ln(\Xi_F)$.

Solution

$$\Xi_F = \sum_{n_1} ... \sum_{n_M} e^{-\beta \sum_{l=1}^{M}(\epsilon_l - \mu) n_l} = \prod_{l=1}^{M} \left(1 + e^{-\beta(\epsilon_l - \mu)}\right) = \prod_{l=1}^{M} \left(1 + z e^{-\beta\epsilon_l}\right) \tag{6}$$

where $z = e^{\beta\mu}$ and each sum is over the possiblities $n_l = 0, 1$ as required for Fermi statistics. We thus find,

$$ln(\Xi_F) = \sum_{l=1}^{M} ln\left(1 + z e^{-\beta\epsilon_l}\right) \tag{7}$$

Quiz Problem 7. Write down the starting expression in the derivation of the grand partition function, Ξ_B for the ideal Bose gas, for a general set of energy levels ϵ_l. Carry out the sums over the energy level occupancies, n_l and hence write down an expression for $ln(\Xi_B)$.

Solution
For the case of Bose statistics the possibilities are $n_l = 0, 1, 2...\infty$ so we find

$$\Xi_B = \sum_{n_1} ... \sum_{n_M} e^{-\beta \sum_{l=1}^{M}(\epsilon_l - \mu) n_l} = \prod_{l=1}^{M} \left(\frac{1}{1 - e^{-\beta(\epsilon_l - \mu)}}\right) = \prod_{l=1}^{M} \left(\frac{1}{1 - z e^{-\beta\epsilon_l}}\right) \tag{8}$$

where the sums are carried out by using the formula for a geometric progression. We thus find,

$$ln(\Xi_B) = -\sum_{l=1}^{M} ln\left(1 - z e^{-\beta\epsilon_l}\right) \tag{9}$$

Quiz Problem 8. (i) Find the single particle energy levels of a non-relativistic quantum particle in a box in 3-d. (ii) Given that

$$ln(\Xi_B) = -\sum_{l} ln(1 - z e^{-\beta\epsilon_l}), \tag{10}$$

using the energies of a quantum particle in a box found in (i), take the continuum limit of the energy sum above to find the inegral form for $ln(\Xi_B)$. Don't forget the ground state term.

Solution
(i) The energy levels of a non-relativistic particle in a 3-d cubic box of size L^3 are,

$$\epsilon_p = \frac{\vec{p}^2}{2m} \quad \text{with} \quad \vec{k} = \frac{\pi}{L}(n_x, n_y, n_z) \tag{11}$$

where $\vec{p} = \hbar\vec{k}$, and n_x, n_y, n_z are integers greater than or equation to one. Hard wall boundaries were assumed.

(ii) Taking the continuum limit we find,

$$ln(\Xi_B) = -\sum_l ln(1 - ze^{-\beta\epsilon_l}) = -\left(\frac{L}{h}\right)^3 \int_0^\infty dp\ 4\pi p^2 ln(1 - ze^{-\beta p^2/2m}) - ln(1 - z) \tag{12}$$

Quiz Problem 9. White dwarf stars are stable due to electron degeneracy pressure. Explain the physical origin of this pressure.

Solution
Even in the ground state, the internal energy of the Fermi gas is positive. This is due to the fact that only one Fermion can be in each energy level so high energy states are occupied at zero temperature. As the density increase, the Fermi energy or energy of the highest occupied state, increases. The pressure is the rate of change of the energy with volume so the pressure increases with the density. This "degeneracy pressure" opposes gravitational collapse and stabilizes white dwarf stars.

Quiz Problem 10. In the condensed phase superfluids are often discussed in terms of a two fluid model. Based on the analysis of the ideal Bose gas, explain the physical basis of the two fluid model.

Solution
The two fluid model considers that the condensed phase is a superfluid while the particles in the excited states behave as a normal fluid. The normal fluid exhibits dissipation and viscosity, while the superfluid has very low values of viscosity and other remarkable properties such as phase coherence.

Quiz Problem 11. Why is the chemical potential of photons in a box, and also acoustic phonons in a crystal, is taken to be zero?

Solution.
The lowest energy state of these systems is zero so any additional photons or phonons may be placed in this state. A more subtle and ultimately the full explanation is through an understanding of the interactions with the reservoir. In the case of massive particles the reservoir is a very large number of the same massive particles so the exchange with the reservoir is through exchange of the same type of particle. In a photon or phonon gas, the reservoir is a system of atoms where the photons or phonons may be absorbed and re-emitted as combinations of different photons or phonons. For this reason the same amount of total free energy in the phonon or photon gas may be divided amongst an arbitrary number of particles, so the chemical potential to add another particle must be zero.

Quiz Problem 12. Derive or write down the blackbody energy density spectrum in three dimensions.
Solution. The blackbody energy density spectrum follows from the equation for the energy of the photon gas in three dimensions,

$$U = 2(\frac{L}{h})^3 \int_0^\infty (\frac{\hbar}{c})^3 d\omega\ 4\pi\omega^2(\hbar\omega)\frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = V\int\ d\omega\ u(\omega) \tag{13}$$

where

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Quiz Problem 13. Write down and explain the relationship between the intensity of radiation emitted by a blackbody (Stefan-Boltzmann law) and the energy density of a photon gas in the blackbody.
Solution. The relationship between the intensity and the energy density of blackbody radiation is,

$$I = \frac{c}{4}\frac{U}{V} = \sigma T^4 \tag{15}$$

The factor $c/4$ is explained as follows: The factor of c converts the energy density of an EM wave into the intensity of radiation crossing a surface whose surface normal is in the same direction as the direction of wave propagation. The factor of $1/4$ has two pieces. First we image that emission from the surface of a blackbody is isotropic so half of the radiation is emitted back into the blackbody. Moreover, the amount of radiation emitted to the exterior is also in all directions on a hemisphere. To find the radiation emitted in the normal direction, we take the component of the electric field in the normal direction, leading to a factor of $cos(\theta)$. However the intensity is the square of the electric field, so it comes with a factor of $cos^2(\theta)$

Quiz Problem 14. Explain the physical origin of the cosmic microwave background (CMB) blackbody spectrum of the universe. It is currently at a temperature of $T_{CMB} = 2.713K$. If the universe is expanding at a constant rate $L(t) = H_0 t$, where H_0 is a constant what is the expected behavior of the temperature $T_{CMB}(t)$.

Solution. During the "photon epoque" of the early universe that is believed to have existed during the period from 10 seconds after the big bang to 377 thousand years after the big bang (that is believed to have occured roughly 13.7 billions years ago), the universe consisted of a gas of charged particles and photons that was equilibrated. At around 380 thousand years after the big bang, Hydrogen and Helium began to form, reducing the scattering of photons and the universe became "transparent". The cosmic microwave background is remnant of the photon gas that existed 380 thousand years ago. Assuming that the photon gas making up the CMB has not changed significantly due to scattering since that time, we can relate the temperature of the CMB to the size of the universe by assuming that the energy in the photon gas is conserved, so that,

$$U = constant = L(t)^3 \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4 \tag{16}$$

where $L(t)$ is the size of the universe.

Quiz Problem 15. Explain the physical origins of the paramagnetic and diamagnetic contributions to the magnetization of the free electron gas.

Solution. The paramagnetic contribution to the magnetization of the free electron gas is the change in the spin polarization due to the application of a magnetic field. The diamagnetic contribution to the magnetization is due to changes in the electron orbitals due to the application of a magnetic field. The diamagnetic contribution can occur even if there is no net spin. To a first approximation, we can add the paramagnetic and diamagnetic contributions. When a paramagnetic contribution occurs, these two contributions are usually of opposite sign.

Quiz Problem 16. Derive or write down the spectral energy density for blackbody radiation in a universe with two spatial dimensions.

Solution. The blackbody energy density spectrum follows from the equation for the energy of the photon gas in two dimensions,

$$U = 2(\frac{L}{h})^2 \int_0^\infty (\frac{\hbar}{c})^3 d\omega \; 2\pi\omega(\hbar\omega)\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = L^2 \int \; d\omega \; u(\omega) \tag{17}$$

Note that here I kept the two polarizations of light even though one of them is along the third direction. We then have,

$$u(\omega) = \frac{\hbar}{\pi c^2}\frac{\omega^2}{e^{\beta\hbar\omega}-1}$$

Quiz Problem 17. Derive or write down the Debye theory for the internal energy for phonons in a square lattice. Derive the low and high temperature limits of the internal energy and specific heat for this system.
Solution. The energy density for the Debye model for the case of a square lattice comes from assuming that phonons are an ideal Bose gas where there is one acoustic mode per atom. The chemical potential is taken to be zero, so we have,

$$U = 2(\frac{L}{h})^2 \int_0^{p_D} dp \; 2\pi p(pv_s)\frac{e^{-\beta pc}}{1-e^{-\beta pv_s}} \tag{19}$$

where,

$$N = (\frac{L}{2\pi})^2 \int_0^{k_D} 2\pi k \; dk, \quad \text{so that} \quad k_D = \left(\frac{4\pi N}{L^2}\right)^{1/2} \tag{20}$$

The factor of two in the front of the energy equation takes into account the fact that there are two phonon modes for the square lattice. This is a rough approximation as only one of the the two modes has the dispersion relation $\epsilon_p = pv_s$.
We define $x = \beta pv_s$, leading to,

$$\frac{U}{L^2} = 4\pi v_s(\frac{1}{h})^2(\frac{1}{\beta v_s})^3 \int_0^{x_D/\beta v_s} dx \; \frac{x^2}{e^x-1} \tag{21}$$

Quiz Problem 18. Find the leading order term in the temperature dependence of the internal energy and specific heat of an three dimensional ultrarelativistic Fermi gas at low temperature.

Solution. The equations for U and N for the three-dimensional ultra-relativistic Fermi gas are,

$$N = 4\pi(\frac{L}{h})^3 \int_0^\infty dp \; p^2 \frac{ze^{-\beta pc}}{1+ze^{-\beta pc}} = \frac{4\pi}{(\beta c)^3}(\frac{L}{h})^3 \int_0^\infty dx \; x^2 \frac{ze^{-x}}{1+ze^{-x}} \tag{22}$$

and

$$U = 4\pi c(\frac{L}{h})^3 \int_0^\infty dp \; p^3 \frac{ze^{-\beta pc}}{1+ze^{-\beta pc}} = \frac{4\pi c}{(\beta c)^4}(\frac{L}{h})^3 \int_0^\infty dx \; x^3 \frac{ze^{-x}}{1+ze^{-x}} \tag{23}$$

We may expand the integral at small z, but this is not useful at low temperature. Instead we carry out the Sommerfeld expansion. Here we write it in more general form, generalizing Eq. (II.73) to,

$$I_s = \int_0^\infty dx \; x^{s-1}\frac{ze^{-x}}{1+ze^{-x}} = \int_0^\infty dx \; x^{s-1}\frac{1}{e^{x-\nu}+1} = \frac{1}{s}\int_0^\infty dx \; x^s \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2} \tag{24}$$

Expanding x^s about ν we have,

$$x^s = (\nu + (x-\nu))^s = f(0) + (x-\nu)f'(0) + \frac{1}{2}(x-\nu)^2 f"(0) + ... \tag{25}$$

where $f(y) = (\nu + y)^s$, so that $f(0) = \nu^s$, $f'(0) = s\nu^{s-1}$, $''(0) = s(s-1)\nu^{s-2}$. so that

$$x^s = (\nu + (x-\nu))^s = \nu^s + (x-\nu)s\nu^{s-1} + \frac{1}{2}(x-\nu)^2 s(s-1)\nu^{s-2} + ... \tag{26}$$

Following the procedure of Eq. (II.77) and (II.78), we then have,

$$I_s = \frac{1}{s}[\nu^s I_0 + s\nu^{s-1} I_1 + \frac{1}{2}s(s-1)\nu^{s-2} I_2 + ...); \quad \text{where} \quad I_n = \int_{-\infty}^{\infty} dt \frac{t^n e^t}{(e^t+1)^2} \tag{27}$$

$I_0 = 1$, while by symmetry I_n is zero for odd n. For even $n > 0$, I_n is related to the Reimann zeta function, through,

$$I_n = 2n(1 - 2^{1-n})(n-1)!\zeta(n), \qquad \text{with} \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945} \tag{28}$$

Since the odd integral $I_0 = 1, \; I_1 = 0, \; I_2 = \pi^2/3$, we find,

$$I_s = \frac{1}{s}[\nu^s + \frac{\pi^2}{6} s(s-1)\nu^{s-2} + ...] \tag{29}$$

Up to a prefactor that is defined differently here, the expansion above is consistent with Eqs. (II.78) and (II.88) as they must be. The expansions we need are then,

$$\frac{N}{V} = \frac{4\pi}{(h\beta c)^3}\frac{1}{3}[\nu^3 + \pi^2\nu + ...] \tag{30}$$

and

$$\frac{U}{V} = \frac{4\pi c}{h^3(\beta c)^4}\frac{1}{4}[\nu^4 + 2\pi^2\nu^2 + ...] \tag{31}$$

The leading order term in the expansion of the chemical potential is found using,

$$\frac{N}{V} = \frac{4\pi}{(h\beta c)^3}\frac{1}{3}(\beta\mu_0)^3 \quad \text{so} \quad \beta\mu_0 = h\beta c\left(\frac{3N}{4\pi V}\right)^{1/3} \tag{32}$$

The next correction is found using,

$$\nu_0^3 = (\nu_1)^3 + \pi^2\nu_0; \qquad \text{so that} \qquad \nu_1 = \nu_0(1 - \frac{\pi^2}{3(\nu_0)^2}) = \beta\epsilon_F[1 - \frac{\pi^2}{3}(\frac{k_B T}{\epsilon_F})^2 + ...]. \tag{33}$$

where $\epsilon_F = \mu_0 = \nu_0/\beta$. The internal energy expansion is,

$$\frac{U}{V} = \frac{\pi c}{h^3(\beta c)^4}\nu^4[1 + \frac{2}{3}\pi^2\nu^{-2} + ...] \approx \frac{\pi c}{h^3(\beta c)^4}\beta\epsilon_F^4[1 - \frac{4\pi^2}{3}(\frac{k_B T}{\epsilon_F})^2 + ...][1 + 2\pi^2\beta\epsilon_F^{-2} + ...] \tag{34}$$

Assigned Problems

Assigned Problem 1. From the density of states for an ideal monatomic gas $\Omega(E)$ given in Eq. (22) of the notes, find the Sackur-Tetrode equation for the entropy, Eq. (23) of the notes.

Solution.

$$\Omega(E) = \frac{2\pi^{1/2}V^N}{N!h^{3N}}\frac{(2\pi mE)^{3N/2-1/2}}{(\frac{3N}{2}-1)!} \tag{35}$$

Using Stirling's approximation and dropping constants, we have

$$k_B ln(\Omega(E)) = k_B N[ln(\frac{V}{h^3}) - ln(N) + 1 - \frac{3}{2}ln(\frac{3N}{2}) + \frac{3}{2} + \frac{3}{2}ln(2\pi mU)] = Nk_B\left[ln[\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}] + \frac{5}{2}\right] \tag{36}$$

Assigned Problem 2. Using the canonical partition function for the ideal gas, show that,

$$(\delta E)^2 = k_B T^2 C_v \tag{37}$$

Solution.
From Eq. (I.131), and using the ideal classical gas expression $Z_N = V^N/(N!\lambda^{3N})$, we have,

$$\delta E^2 = \frac{\partial^2 ln(Z)}{\partial \beta^2} = \delta E^2 = \frac{\partial^2}{\partial \beta^2}[ln(\frac{V^N(2\pi m)^{3N/2}}{N!h^{3N}} - \frac{3N}{2}ln(\beta)] = \frac{3}{2}N(k_B T)^2 = k_B T^2 C_V \tag{38}$$

where $C_V = 3Nk_B/2$ for the classical monatomic non-relativistic ideal gas in three dimensions.

Assigned Problem 3. Using the grand partition function of the ideal classical gas show that,

$$(\delta N)^2 = N k_B T \rho \kappa_T \tag{39}$$

Solution. From Eq. (I.135), and using the expression for the grand partition function for the classical gas (Eq. (II.25)),

$$(\delta N)^2 = (k_B T)^2 \frac{\partial^2 ln(\Xi)}{\partial \mu^2} = (k_B T)^2 \frac{V}{\lambda^3 \beta^2} e^{\beta\mu} = \frac{V}{\lambda^3} e^{\beta\mu} = \frac{PV}{k_B T} = N \tag{40}$$

where we used (II.25) to write $\alpha z = ln(\Xi) = PV/k_B T$. Also the right hand side of Eq. (17) for the classical ideal gas is,

$$N k_B T \rho \kappa_T = k_B T \frac{N^2}{V} \kappa_T = k_B T \frac{N^2}{PV} = N \tag{41}$$

where we used Eq. (II.17) for κ_T for the ideal gas

Assigned Problem 4. At high temperatures we found that the ideal quantum gases reduce to the ideal classical gases. Derive the next term in the expansion of the equation of state of the ideal Fermi gas at high temperatures, and verify that,

$$\frac{PV}{Nk_B T} = 1 + \frac{1}{4\sqrt{2}} \frac{\lambda^3 N}{V} + ... \qquad \text{Fermi gas} \tag{42}$$

The pressure of the ideal Fermi gas is higher than that of the classical gas at the same temperature and volume. Why? Carry out a similar expansion for the Bose gas. Is the pressure higher or lower than the ideal classical gas at the same values of T, V? Why?

Solution. For the Bose case, expanding to second order gives

$$\frac{N\lambda^3}{V} = g_{3/2}(z) = z + \frac{z^2}{2\sqrt{2}} \approx z_1 + \frac{z_0^2}{2\sqrt{2}} \tag{43}$$

where $z_0 = N\lambda^3/V$ is the leading order solution, we then find,

$$z_1 = z_0 - \frac{z_0^2}{2\sqrt{2}} \tag{44}$$

We substitute this into the second order expansion of the equation of state,

$$\frac{P}{k_B T} = \frac{1}{\lambda^3}[z + \frac{z^2}{4\sqrt{2}} + ...] \approx \frac{1}{\lambda^3}[z_1 + \frac{z_1^2}{4\sqrt{2}} + ...] \approx \frac{1}{\lambda^3}[z_0 - \frac{z_0^2}{2\sqrt{2}} + (\frac{(z_0 - \frac{z_0^2}{2\sqrt{2}})^2}{4\sqrt{2}} + ...] \tag{45}$$

keeping terms to order z_0^2 gives,

$$\frac{P}{k_B T} = \frac{z_0}{\lambda^3}[1 - z_0(\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}}) + ...] \tag{46}$$

substituting $z_0 = N\lambda^3/V$ gives,

$$\frac{PV}{Nk_B T} = 1 - \frac{1}{4\sqrt{2}}\frac{\lambda^3 N}{V} + ... \qquad \text{Bose gas} \tag{47}$$

Analysis for the Fermi gas is the same, except that the sign on the correction term is positive. The Bose gas has reduced pressure as compared to the classical gas at the same temperature, while the Fermi gas has higher pressure than the classical case. This expansion can be extended to higher order and in general is written as,

$$\frac{PV}{Nk_B T} = \sum_{l=1}^{\infty} a_l \alpha^{l-1}; \qquad \text{where} \qquad \alpha = \frac{\lambda^3 N}{V} \tag{48}$$

This expansion is valid when α is small, which means low density and/or high temperatures. In general expansions of this type are called **virial expansions** and have played an important role in characterizing interactions in gases. In classical gases the second virial coefficient a_2 is determined by the strength of the pair interactions, as we shall see in Part 3 of the course. Here the terms $l > 1$ are due to quantum effects. In real quantum gases, both quantum effects and interactions can be important. Recall than our condition for quantum effects to be important was that the interparticle spacing $L_c < \lambda$. When this is true α is significant and more terms are required in the virial expansion.

Assigned Problem 5. By expanding the denominator of the integral, $1/(1+y)$ for small $y = ze^{-x^2}$ show that,

$$f_{3/2}(z) = \frac{4}{\pi^{1/2}}\int_0^{\infty} dx\ x^2 \frac{ze^{-x^2}}{1+ze^{-x^2}} = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}z^l}{l^{3/2}} \tag{49}$$

Solution.
We use the expansion

$$\frac{1}{1+y} = (1 - y + y^2 - y^3...); \qquad \text{and} \qquad \int_0^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}} \tag{50}$$

so that,

$$f_{3/2}(z) = \frac{4}{\pi^{1/2}}\int_0^{\infty} dx\ x^2 \frac{ze^{-x^2}}{1+ze^{-x^2}} = \sum_{l=1}^{\infty}(-1)^{l+1}z^l \int_0^{\infty} dx\ x^2 e^{-lx^2} = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}z^l}{l^{3/2}} \tag{51}$$

Assigned Problem 6. Derive expressions for Z, Ξ, PV, μ and U for the classical gas in one and two dimensions. Are the results what you expect? How do they compare with the result in three dimensions. White general expressions that are valid in any dimension.

Solution In d-dimensions, the partition functions are,

$$Z = \frac{L^{dN}}{\lambda^{dN}N!}, \qquad \Xi = e^{\alpha z} \qquad \alpha = (\frac{L}{\lambda})^d \tag{52}$$

$$F = -k_B T ln(Z) = -k_B T ln(\frac{L^{dN}}{\lambda^{dN}N!}) \tag{53}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{L^d,N} = k_B ln(\frac{L^{dN}}{\lambda^{dN}N!}) + \frac{d}{2}Nk_B \tag{54}$$

The internal energy is found by combining (31) and (32), so that,

$$U = F + TS = \frac{d}{2} N k_B T \tag{55}$$

The pressure is given by,

$$P = -\left(\frac{\partial F}{\partial L^d}\right)_{T,N} = k_B T \frac{N}{L^d} = \frac{k_B N T}{L^d}, \tag{56}$$

which is the ideal gas law, while the chemical potential is,

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,L^d} = k_B T ln(\lambda^d N/L^d) \tag{57}$$

Assigned Problem 7. Derive expressions for Ξ, PV, N/V and U for the Bose gas in one and two dimensions. White general expressions that are valid in any dimension. Find the leading order terms in the high temperature expansions for these quantities. Are the results what you expect? How do they compare with the result in three dimensions and with the classical behavior.

Solution. For a Bose gas with dispersion relation $\epsilon_p = p^2/2m$ in d dimensions

$$\frac{P}{k_B T} = \frac{1}{\lambda^d} g_{d/2+1}(z) - \frac{1}{L^d} ln(1-z); \qquad \frac{N}{L^d} = \frac{1}{\lambda^d} g_{d/2}(z) - \frac{1}{L^d}\frac{z}{1-z}; \qquad \frac{U}{L^d} = \frac{d}{2}\frac{k_B T}{\lambda^d} g_{d/2+1}(z) \tag{58}$$

These results are found from the integral forms in one and two dimensions below along with the three dimensional result derived in lectures.

$$ln(\Xi_B) = -\frac{L}{2\pi\hbar}\int_0^\infty 2dp ln(1 - ze^{-\beta p^2/2m}) - ln(1-z); \quad 1 - dimension \tag{59}$$

and

$$ln(\Xi_B) = -(\frac{L}{2\pi\hbar})^2 \int_0^\infty 2\pi p dp ln(1 - ze^{-\beta p^2/2m}) - ln(1-z) \quad 2 - dimension \tag{60}$$

A series expansion as carried out for these integrals and the similar forms for N/V and U lead to the results above. To leading order in the fugacity of the equation for N/V, we find the chemical potential to be the same as that of the ideal classical gas in d dimensions, i.e. $z = N(\lambda/L)^d$. The dimensional dependence comes from the different powers of the factor $(L/h)^d$, and the factors of p in the integral. The integrals that are needed are,

$$\int_0^\infty x e^{-x^2 l} dx = \frac{1}{2l}; \qquad \int_0^\infty e^{-x^2 l} dx = \frac{1}{2}(\frac{\pi}{l})^{1/2} \tag{61}$$

We then have,

$$\frac{PL}{k_B T} = ln(\Xi_B) = \frac{L}{2\pi\hbar}(\frac{2m}{\beta})^{1/2} \sum_{l=1}^{\infty} \frac{z^l}{l} \int_0^\infty 2dx e^{-x^2 l} \quad - \; ln(1-z); \quad 1 - dimension \tag{62}$$

and

$$\frac{PL^2}{k_B T} = ln(\Xi_B) = (\frac{L}{2\pi\hbar})^2 (\frac{2m}{\beta}) \sum_{l=1}^{\infty} \frac{z^l}{l} \int_0^\infty 2\pi x dx e^{-x^2 l} \quad - \; ln(1-z); \quad 2 - dimensions \tag{63}$$

which reduce to the expression given in Eq. (36) for $P/(k_B T)$.

Assigned Problem 8. Derive expressions for Ξ, PV, N/V and U for the Fermi gas in one and two dimensions. White general expressions that are valid in any dimension. Find the leading order terms in the high temperature expansions for these quantities. Are the results what you expect? How do they compare with the result in three dimensions, and with the classical gas.

Solution The relations for the non-relativistic Fermi gas are,

$$\frac{P}{k_BT} = \frac{1}{\lambda^d} f_{d/2+1}(z); \qquad \frac{N}{L^d} = \frac{1}{\lambda^d} f_{d/2}(z); \qquad \frac{U}{L^d} = \frac{d}{2}\frac{k_BT}{\lambda^d} f_{d/2+1}(z) \tag{64}$$

These results are found from the integrals,

$$ln(\Xi_F) = \frac{L}{2\pi\hbar}\int_0^\infty 2dpln(1+ze^{-\beta p^2/2m}); \quad 1-dimension \tag{65}$$

and

$$ln(\Xi_F) = (\frac{L}{2\pi\hbar})^2\int_0^\infty 2\pi pdpln(1+ze^{-\beta p^2/2m}) \quad 2-dimension \tag{66}$$

These integrals and the analogous equations for N and V are expanded as in the three dimensional case. Following the procedure given in the solution to problem 7, we have,

$$\frac{PL}{k_BT} = ln(\Xi_F) = \frac{L}{2\pi\hbar}(\frac{2m}{\beta})^{1/2}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}z^l}{l}\int_0^\infty 2dxe^{-x^2l}; \quad 1-dimension \tag{67}$$

and

$$\frac{PL^2}{k_BT} = ln(\Xi_F) = (\frac{L}{2\pi\hbar})^2(\frac{2m}{\beta})\sum_{l=1}^{\infty}\frac{(-1)^{l+1}z^l}{l}\int_0^\infty 2\pi xdxe^{-x^2l}; \quad 2-dimensions \tag{68}$$

which reduce to the expression for P/k_BT given in Eq. (42). The leading order expansion of $f_{d/2}(z)$ at high temperature gives the chemical potential of the ideal classical gas in d dimensions so we recover the classical gas in d dimensions at sufficiently high temperatures.

Assigned Problem 9. Using the results of Problem 7, discuss the behavior of the Bose gas at low temperatures in one and two dimensions. Is a finite temperature Bose condensation predicted ? Explain your reasoning.

Solution. At at any temperature, the chemical potential potential of the ideal non-relativistic Bose gas in dimensions less than $2+\delta$ cannot be one in the thermodynamic limit, as,

$$g_n(z) = \sum_l \frac{z^l}{l^n} \tag{69}$$

diverges for $z=1$ and $n \le 1$. Since z cannot approach one, the term $z/(V(1-z))$ approaches zero in the thermodynamic limit, indicating that it is impossible for a finite fraction of the particles to be in the ground state. There is therefore no Bose condensation at finite temperature in one and two dimensional ideal non-relativistic Bose gases.

Assigned Problem 10. Discuss the behavior of the Fermi gas at zero temperatures in one and two dimensions. Is there different behavior as a function of dimension? Explain your reasoning.

Solution. We calculate the energy and degeneracy pressure to see if there is a dependence on dimension. We only carry out the ground state calculation. The Fermi wavevector in one, two and three dimensions is given by,

$$k_{F1} = \frac{\pi N}{L}; \quad k_{F2} = (\frac{4\pi N}{L^2})^{1/2}; \qquad k_{F3} = (\frac{6\pi^2 N}{V})^{1/3} \tag{70}$$

so the Fermi energy is given by,

$$\epsilon_{F1} = \frac{\hbar^2}{2mL^2}(\pi N)^2; \qquad \epsilon_{F2} = \frac{\hbar^2}{2mL^2}(4\pi N); \quad \epsilon_{F2} = \frac{\hbar^2}{2mL^2}(6\pi^2 N)^{2/3}; \tag{71}$$

The internal energy is given by,

$$U = (\frac{L}{2\pi})^d \int d^d k \frac{\hbar^2 k^2}{2m} \tag{72}$$

In one two and three dimensions we find,

$$U_1 = \frac{1}{3} N E_{F1}; \quad U_2 = \frac{1}{2} N E_{F2}; \quad U_3 = \frac{3}{5} N E_{F3} \tag{73}$$

The degeneracy pressure is given by,

$$P = -\left(\frac{\partial U}{\partial L^d}\right)_{N,T} \tag{74}$$

Since E_F is proportional to $1/L^2$ doing the derivative with respect to L, L^2 and L^3 in one two and three dimensions, leads to the following expressions for the degeneracy pressure,

$$P_1 = \frac{2}{3L} N E_{F1}; \quad P_2 = \frac{1}{2L^2} N E_{F2}; \quad U_3 = \frac{2}{5L^3} N E_{F3}. \tag{75}$$

For fixed number of particles, the degeneracy pressure and internal energy are much higher for the one and two dimensional cases, first because E_F grows much more rapidly with N as the dimension is reduced and second because the prefactor grows more slowly with L as the dimension is reduced. This results are expected as particles are more confined in one and two dimensions, so the effect of Pauli exchange is stronger, so the total energy is expected to grow more rapidly in lower dimension and the degeneracy pressure should be higher.

Assigned Problem 11. For the 3-D non-relativistic case: a) Find the entropy of the ideal Bose gas in the condensed phase $T < T_c$. b) Find the entropy of the ideal Fermi gas at low temperatures to leading order in the temperature. Does the Fermi or Bose gas have higher entropy at low temperatures? Why?

Solution. Using the thermodynamic relation, $U = TS - PV + \mu N$, we find,

$$TS = U + PV - \mu N = \frac{1}{2} PV - \mu N \tag{76}$$

For the Bose gas at $T < T_c$ where $\mu = 0$, we have $P/k_B T = \zeta(5/2)/\lambda^3$ and using $PV = 2U/3$ we find,

$$TS = \frac{5}{2} PV = \frac{5}{2} \frac{k_B T V}{\lambda^3} \zeta(5/2) \tag{77}$$

For the Fermi gas we have,

$$TS = \frac{5}{2} PV - \mu N \tag{78}$$

Using the results (86) and (92) from the lecture notes we find to first order,

$$TS = N\epsilon_F [1 + \frac{5}{12}\pi^2 \left(\frac{k_b T}{\epsilon_F}\right)^2] - N\epsilon_F [1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2] = N\epsilon_F \frac{\pi^2}{2}\left(\frac{k_b T}{\epsilon_F}\right)^2 \tag{79}$$

In both the Bose and Fermi cases, the entropy approaches zero at low temperature. In the Bose case $S \propto T^{3/2}$, while in the Fermi case $S \propto T$, so the Bose entropy approaches zero more rapidly at low enough temperatures. To find the temperature at which the two entropies are equation we write,

$$N\epsilon_F \frac{\pi^2}{2} \frac{k_B T*}{\epsilon_F^2} = \frac{5}{2} \frac{(k_B T*)^{3/2} V}{(\lambda/(k_B T))^3} \zeta(5/2); \qquad \text{so that} \qquad (k_B T^*)^{1/2} = \frac{2}{5} (\frac{\lambda}{k_B T})^3 [N\epsilon_F \frac{\pi^2}{2\epsilon_F^2 V \zeta(5/2)}] \tag{80}$$

For temperatures lower than T^*, the Bose gas has lower entropy.

Assigned Problem 12. Show that a $d-dimensional$ Bose gas with dispersion relation $\epsilon_p = cp^s$ obeys the relation,

$$P = \frac{s}{d}\frac{U}{L^d} \tag{81}$$

Solution. The equations for a Bose gas with this dispersion relation in d dimensions is written as,

$$\frac{P}{k_BT} = -\frac{c_d}{h^d}\int_0^\infty dp\ p^{d-1}ln(1-ze^{-\beta cp^s}); \qquad \frac{U}{L^d} = \frac{c_d}{h^d}\int_0^\infty dp\ p^{d-1}(cp^s)\frac{ze^{-\beta cp^s}}{1-ze^{-\beta cp^s}} \tag{82}$$

Integrating the pressure equation by parts gives,

$$I_1 = -\int_0^\infty dp\ p^{d-1}ln(1-ze^{-\beta cp^s}) = -\frac{1}{d}p^d ln(1-ze^{-\beta cp^s})|_0^\infty + \frac{s\beta}{d}\int_0^\infty dp\ p^{d-1}(cp^s)\frac{ze^{-\beta cp^s}}{1-ze^{-\beta cp^s}} \tag{83}$$

The first term on the right hand side is zero and comparison of the remaining integral with the energy equation proves relation (81).

Assigned Problem 13. By expanding Eq. (II.103) of the lecture notes to fourth order in y, show that it reproduces $-f_R(y)$ as given in Eq. (II.111).

Solution. An expansion to fourth order in y of $f(y)$ gives,

$$f(y) = \frac{-N}{4\beta J}y^2 + N[ln(2) + ln(cosh(y))] \approx -N[y^2(\frac{1}{4\beta J} - \frac{1}{2}) + \frac{y^4}{12}] \tag{84}$$

or in its more usual form,

$$\frac{-4\beta J f(y)}{N} = (1-2\beta J)y^2 + \frac{4\beta J y^4}{12} \tag{85}$$

a change in sign of the prefactor of the y^2 term indicates the location of the phase transition. Minimization with respect to y leads to the order parameter equation and solving gives the behavior near the critical point.

Assigned Problem 14. Write down the 4×4 transfer matrix for two ising chains connected together. Show that this transfer matrix can be written in the form,

$$\hat{T} = (\hat{M}_2)(\hat{M}_1 \otimes \hat{M}_1) \tag{86}$$

where $\hat{M}_1$ is the transfer matrix of the Ising chain, while $\hat{M}_2$ is a 4×4 diagonal matrix. $\otimes$ is the direct or Kronecker product.

Solution. The transfer matrix for the system has matrix elements,

$$T_{S,S'} = e^{2KS_1S_2 + KS_1S_1' + KS_2S_2'}; \qquad K = \beta J \tag{87}$$

One approach to this is write down the transfer matrix above and compare it with the matrix $A = (\hat{M}_1 \otimes \hat{M}_1)$. You will find that $T = BA$ where B is a diagonal matrix with diagonal elements $(e^{-2K}, e^{2K}, e^{2K}, e^{-2K})$.

Assigned Problem 15. Use the transfer matrix method to find the partition function for the one dimensional Ising model in a magnetic field h. From your expression find the magnetic susceptibility at $h = 0$. Does it obey a Curie Law? Here we have absorbed μ_s into h.

Solution. The transfer matrix has elements

$$T_{11} = ab; \quad T_{12} = T_{21} = 1/a; \quad T_{2,2} = a/b; \qquad a = e^{\beta J};\ b = e^{\beta h} \tag{88}$$

The characteristic equation is then,

$$(ab - \lambda)(\frac{a}{b} - \lambda) - (\frac{1}{a})^2 = 0 \tag{89}$$

which has solutions,

$$\lambda_{\pm} = \frac{1}{2}[acosh(\beta h) \pm \left([acosh(\beta h)]^2 + 8sinh(2\beta J)\right)^{1/2}] \tag{90}$$

we then use,

$$Z = Nln(\lambda_+); \qquad m = \frac{1}{N}\frac{\partial ln(Z)}{\partial(\beta h)}; \qquad \chi = \frac{\partial m}{\partial h} \tag{91}$$

to find,

$$m = \frac{sinh(\beta h)}{[e^{-4\beta J} + sinh^2(\beta h)]^{1/2}} \tag{92}$$

so that

$$\chi = \frac{\beta cosh(\beta h)}{[e^{-4\beta J} + sinh^2(\beta h)]^{1/2}} - \frac{\beta}{2}\frac{2Sinh^2(\beta h)Cosh(\beta h)}{[e^{-4\beta J} + sinh^2(\beta h)]^{3/2}} \tag{93}$$

the zero field susceptibility comes from the first term, which gives (putting the μ_s^2 term back in,

$$\chi_0(T \to 0) = \frac{\mu_s^2}{k_B T} e^{2J/k_B T} \tag{94}$$

It has a Curie law part, but the dominant part is the exponential term. This is again due to the fact that there is a gap in the energy spectrum for the Ising system.

Assigned Problem 16. Find the thermodynamic properties, PV, U, S, C_V, N of a photon gas in d dimensions. Show that the entropy per photon is independent of temperature.

Solution. We use the relations,

$$N = 2\left(\frac{L}{2\pi\hbar}\right)^d \int_0^\infty c_d p^{d-1} dp\ \frac{e^{-\beta pc}}{1 - e^{-\beta pc}} = 2c_d \left(\frac{L}{h}\right)^d (\frac{1}{\beta c})^d \int_0^\infty dx\ \frac{x^{d-1}}{e^x - 1} \tag{95}$$

and

$$U = 2\left(\frac{L}{2\pi\hbar}\right)^d \int_0^\infty c_d p^{d-1} dp\ (pc)\frac{e^{-\beta pc}}{1 - e^{-\beta pc}} = 2c_d c \left(\frac{L}{h}\right)^d (\frac{1}{\beta c})^{d+1} \int_0^\infty dx\ \frac{x^d}{e^x - 1} \tag{96}$$

along with $PV = sU/d$, with $s = 1$ and the integral,

$$\int_0^\infty \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \tag{97}$$

to find,

$$U = 2c_d c d! \zeta(d+1)\left(\frac{L}{h}\right)^d (\frac{k_B}{c})^{d+1} T^{d+1}; \qquad N = 2c_d (d-1)! \zeta(d) \left(\frac{L}{h}\right)^d (\frac{k_B}{c})^d T^d \tag{98}$$

We also have,

$$TS = U + PV - \mu N = (d+1)U/d \propto T^{d+1} \tag{99}$$

From this it is evident that both S and N are proportional to T^d, so S/N is temperature independent. Also,

$$C_V = \frac{\partial U}{\partial T} = 2(d+1)!\zeta(d+1)c_d c \left(\frac{L}{h}\right)^d (\frac{k_B}{c})^{d+1} T^d \tag{100}$$

Assigned Problem 17. Find the thermodynamic properties, U and C_V for the Debye phonon model in d dimensions.

Solution. We use the relation,

$$U = d\left(\frac{L}{2\pi\hbar}\right)^d \int_0^{p_d} c_d p^{d-1} dp\ (pv_s)\frac{e^{-\beta p v_s}}{1-e^{-\beta p v_s}} = d c_d v_s \left(\frac{L}{h}\right)^d (\frac{k_B}{v_s})^{d+1} T^{d+1} \int_0^{x_D} dx\ \frac{x^d}{e^x - 1} \tag{101}$$

where the factor of d in front ensures that we recover the high temperature equipartition result. $x_D = \beta p_D v_s$ and we define the Debye temperature through $k_B T_D = p_D v_s$, so that $x_D = \beta k_B T_D = T_D/T$. We also use the integral,

$$\int_0^\infty \frac{x^{s-1}dx}{e^x - 1} = \Gamma(s)\zeta(s), \tag{102}$$

to find that at low temperatures $T_D/T \to \infty$, so the behavior is like that of the photon gas in d dimensions, with $c \to v_s$, and multiplied by $d/2$ due to the difference in degeneracy (2 for photons, d for phonons). We then have ,

$$U = d c_d v_s d!\zeta(d+1)\left(\frac{L}{h}\right)^d (\frac{k_B}{v_s})^{d+1} T^{d+1}; \quad C_V = \frac{\partial U}{\partial T} = d(d+1)!\zeta(d+1)c_d v_s \left(\frac{L}{h}\right)^d (\frac{k_B}{v_s})^{d+1} T^d \tag{103}$$

At high temperatures, the behavior is like that of a classical gas in a harmonic potential so that $U = dNk_BT, C_V = Ndk_B$, and $PV = 2U/d$.

Assigned Problem 18. Consider a two dimensional electron gas in a magnetic field that is strong enough so that all of the particles can be accomodated in the lowest Landau level. Taking into account both the paramagnetic and diamagnetic contributions, find the magnetization at temperature $T = 0K$.

Solution. Since we are in the ground state we can consider just the energy and find the magnetization using

$$M = -\frac{\partial E_G}{\partial B} \tag{104}$$

This is true as

$$M = \frac{1}{\beta}\frac{\partial ln(Z)}{\partial B} = -\frac{\partial F}{\partial B} \to -\frac{\partial E_G}{\partial B} \tag{105}$$

The lowest Landau level is split by the application of a magnetic field into two sublevels with energies,

$$E_- = \frac{1}{2}\hbar\omega_c - \mu_s B; \quad E_+ = \frac{1}{2}\hbar\omega_c + \mu_s B \tag{106}$$

We can consider the two contributions to the magnetization separately. The diamagnetic contribution comes from the diamagnetic part of the energy $N\hbar\omega_c/2$, so

$$M_D = -\frac{\partial}{\partial B}(N\frac{\hbar e B}{2m}) = -N\frac{e\hbar}{2m} = -N\mu_B \quad \text{where} \quad \mu_B = \frac{e\hbar}{2m_e} \tag{107}$$

where μ_B is the Bohr magneton and the spin magnetic moment of the electron is approximately μ_B.

The paramagnetic contribution to the magnetization comes from the filling of the up and down levels as a function of applied field B. If there are N electrons, the first field (B_0) at which this number of electrons can be accomodated in the lowest Landau level is found from the relation $2g = N$, so we have,

$$2g = 2\frac{B_0 Ae}{h} = N; \quad or \quad B_0 = \frac{h}{2e}\frac{N}{A} \tag{108}$$

where A is the area of the sample. The paramagnetic (spin) contribution to the magnetization is $M_P = \mu_B(N_\uparrow - N_\downarrow)$. At B_0, $N_\uparrow = N_\downarrow$, so $M = 0$. If the magnetic field is increased further, the degeneracy of the ground state continues to increase linearly with the field. There is a second field at which all of the electrons can be accomodated in the lowest Zeeman level (up spin sublevel). This field is given by,

$$B_1 = \frac{h}{e}\frac{N}{A} \tag{109}$$

for fields greater than this field all of the spins are in the up spin state, so $M_P = N\mu_B$. We then find,

$$M = M_D + M_P = -N\mu_B + N\mu_B = 0 \quad \text{for} \quad B > B_1 \tag{110}$$

In the field range $B_0 < B < B_1$, the number of spins in the up spin sublevel is g, while the number in the down spin sublevel is $N - g$, so the paramagnetic contribution is,

$$M_P = (g - (N - g))\mu_B = (2g - N)\mu_B = (2\frac{BAe}{h} - N)\mu_B \tag{111}$$

so the magnetization in this regime is,

$$M = M_D + M_P = (2\frac{BAe}{h} - N)\mu_B - N\mu_B = (2\frac{BAe}{h} - 2N)\mu_B \quad B_0 < B < B_1 \tag{112}$$

The magnetic response is diamagnetic for $B < B_1$, while there is no magnetic moment for $B > B_1$, at least within this approximation.

Assinged Problem 19. Show that the low temperature specific heat of the relativistic Fermi gas in three dimensions is given by,

$$\frac{C_V}{Nk_B} = \frac{\pi^2(x^2+1)^{1/2}}{x^2}\frac{k_B T}{m_0 c^2}; \quad \text{where} \quad x = p_F/(m_0 c) \tag{113}$$

Solution. Optional.

PHY831 - Quiz 3, Monday October 10 2011

Answer all questions. Time for quiz - 20 minutes

Name:

1. Write down the equation for the thermal de Broglie wavelength. Explain its importance in the study of classical and quantum gases.

Solution The thermal de Broglie wavelength, which is really a thermal length not a wavelength, is given by,

$$\lambda_T = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}. \tag{114}$$

When the thermal length is small, thermal fluctuations in momentum and kinetic energy of gas particles occur on short length and time scales. If this length is shorter than the interparticle spacing $L_c = (V/N)^{1/3}$, then thermal fluctuations dominate quantum effects, so in the regime $\lambda_T << L_c$ the gas behaves like a classical gas.

2. (i) Find or write down the single particle energy levels of a non-relativistic quantum particle in a box in 3-d. (ii) Given that

$$ln(\Xi_B) = -\sum_l ln(1 - ze^{-\beta\epsilon_l}), \tag{115}$$

using the energies of a quantum particle in a box found in (i), take the continuum limit of the energy sum above to find the integral form for $ln(\Xi_B)$. Don't forget the ground state term.

Solution

(i) The energy levels of a particle in a box with hard walls are given by,

$$\epsilon_p = \frac{p^2}{2m}; \qquad \text{where} \qquad \vec{p} = \frac{\hbar\pi}{L}(n_x, n_y, n_z) \tag{116}$$

where n_x, n_y, n_z have positive integer values. The continuum limit is then,

$$ln(\Xi_B) = -\sum_p ln(1 - ze^{-\beta\epsilon_p}) = -\left(\frac{L}{h}\right)^3 \int_0^\infty 4\pi p^2 ln(1 - ze^{-\beta p^2/2m}) - ln(1-z) \tag{117}$$

where $z = e^{\beta\mu}$, and where $ln(1-z)$ is the extra ground state contribution.

3. Consider an ultrarelativistic Fermi gas with the relation $\epsilon_p = pc$ in a box of volume V. Derive or write down the integral forms of PV, N and U for this gas.

Solution

The only change required is to replace $\epsilon_p = p^2/2m$ by $\epsilon_p = pc$ for in the Fermi relations of Eq. (II.44-46) , so we have,

$$\frac{PV}{k_B T} = ln(\Xi_F) = \sum_p ln(1 + ze^{-\beta\epsilon_p}) = \left(\frac{L}{h}\right)^3 \int_0^\infty 4\pi p^2 ln(1 + ze^{-\beta pc}), \tag{118}$$

$$N = \sum_p < n_p > = \sum_p \frac{ze^{-\beta\epsilon_p}}{1 + ze^{-\beta\epsilon_p}} = \left(\frac{L}{h}\right)^3 \int_0^\infty 4\pi p^2 \frac{ze^{-\beta pc}}{1 + ze^{-\beta pc}}, \tag{119}$$

and

$$U = \sum_p \epsilon_p < n_p > = \sum_p (\epsilon_p) \frac{ze^{-\beta\epsilon_p}}{1 + ze^{-\beta\epsilon_p}} = \left(\frac{L}{h}\right)^3 \int_0^\infty 4\pi p^2 (pc) \frac{ze^{-\beta pc}}{1 + ze^{-\beta pc}}. \tag{120}$$

Note that for the relativistic case U is NOT equal to $3PV/2$ in three dimensions, however by integrating Eq. (110) by parts it is seen that for the case $\epsilon = pc$, we have $U = 3PV$.

PHY831 - Quiz 4, Wednesday October 19 2011
Answer all questions. Time for quiz - 20 minutes

Name:

1. Explain the physical origin of the cosmic microwave background (CMB) blackbody spectrum of the universe. It is currently at a temperature of $T_{CMB} = 2.713K$. If the universe is expanding at a constant rate $L(t) = H_0 t$, where H_0 is a constant, what is the expected behavior of the temperature $T_{CMB}(t)$. The relation $\frac{U}{V} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4$ may be useful.

Solution. During the "photon epoque" of the early universe is believed to have existed during the period from 10 seconds after the big bang to 377 thousand years after the big bang (the big bang occured roughly 13.7 billions years ago), the universe consisted of a gas of charged particles and photons that was equilibrated. At around 377 thousand years after the big bang, Hydrogen and Helium began to form, reducing the scattering of photons and the universe became "transparent". The cosmic microwave background is remnant of the photon gas that existed 377 thousand years ago. Assuming that the photon gas making up the CMB has not changed significantly due to scattering since that time, we can relate the temperature of the CMB to the size of the universe by assuming that the energy in the photon gas is conserved, so that,

$$U = constant = L(t)^3 \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4 \approx (H_0 t)^3 \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4 \tag{121}$$

where $L(t)$ is the size of the universe. The temperature of the CMB then behaves as,

$$T(t) = \left(\frac{15\hbar^3 c^3 U}{\pi^2 k_B^4 H_0^3}\right)^{1/4} t^{-3/4} \tag{122}$$

2. Explain the physical origins of the paramagnetic and diamagnetic behaviors of the free electron gas.

Solution. The paramagnetic contribution to the magnetization of the free electron gas is due to the change in the spin polarization due to the application of a magnetic field. The diamagnetic contribution to the magnetization is due to changes in the electron orbital contribution to the magnetization due to the application of a magnetic field. The diamagnetic contribution can occur even if there is no net spin. To a first approximation, we can add the paramagnetic and diamagnetic contributions. When a paramagnetic contribution occurs, these two contributions are usually of opposite sign, with the paramagnetic susceptibility in the same direction as the applied field, while the diamagnetic component opposes the applied field due to Lenz's law.

3. For systems with dispersion relation $\epsilon_p = cp^s$, for a $d - dimensional$ Fermi gas, show that $PL^d = \frac{s}{d}U$. To show this, you can write the integral,

$$\int d^d p = \int c_d p^{d-1} dp \tag{123}$$

without having to know the expression for c_d.

Solution. The equations for a Fermi gas with this dispersion relation in d dimensions may be written as,

$$\frac{P}{k_B T} = \frac{c_d}{h^d} \int_0^\infty dp\, p^{d-1} ln(1 + ze^{-\beta c p^s}); \qquad \frac{U}{L^d} = \frac{c_d}{h^d} \int_0^\infty dp\, p^{d-1} (cp^s) \frac{ze^{-\beta c p^s}}{1 + ze^{-\beta c p^s}} \tag{124}$$

Integrating the pressure equation by parts gives,

$$I_1 = \int_0^\infty dp\, p^{d-1} ln(1 + ze^{-\beta c p^s}) = \frac{1}{d} p^d ln(1 + ze^{-\beta c p^s})|_0^\infty + \frac{s\beta}{d} \int_0^\infty dp\, p^{d-1} (cp^s) \frac{ze^{-\beta c p^s}}{1 + ze^{-\beta c p^s}} \tag{125}$$

The first term on the right hand side is zero and comparison of the remaining integral with the energy equation proves that $PL^d = \frac{s}{d}U$.