

Solutions to problems for Part 2

Sample Quiz Problems

Quiz Problem 1. Write down the equation for the thermal de Broglie wavelength. Explain its importance in the study of classical and quantum gases.

Solution

$$\lambda = \frac{h}{\sqrt{2\pi mk_B T}} \quad (1)$$

This is of the form h/p_T , where $p_T = (2\pi mk_B T)^{1/2}$ is an average thermal momentum. Define the average interparticle spacing of a gas $L_c = (V/N)^{1/3}$. If $\lambda > L_c$ quantum effects become important in the thermodynamics.

Quiz Problem 2. Why are the factors $1/N!$ and $1/h^{3N}$ introduced into the derivation of the partition function of the ideal classical gas?

Solution

The factor $1/N!$ is needed to account for the fact that when an integration is carried out over all phase space for N particles, all permutations of the particle identities is included. For indistinguishable particles this must be removed. The factor $1/h^{3N}$ takes account of the Heisenberg uncertainty principle which states that the smallest phase space volume that makes sense is $(\hbar/2)^3$. The fact that it is $1/h^3$ instead of $1/(\hbar/2)^3$ for each particle is to reproduce the high temperature behavior of quantum gases.

Quiz Problem 3. Explain why the heat capacity at constant volume of a nitrogen molecule is roughly $1.5Nk_B$ at $1K$ but $2.5Nk_B$ at room temperature. What are the expressions for the heat capacity at constant pressure for these two cases?

Solution

At low temperatures only the translational degrees of freedom are active so we have $k_B T/2$ of energy for each of three degrees of freedom. The specific heat is then $3k_B T N/2$. Once the temperature is larger than the spacing between rotational energy levels $\hbar^2/2I$ where I is the moment of inertia, then the rotational degrees of freedom are also active as we get two more degrees of freedom. Only two rotational degrees of freedom contribute as the moment of inertia about the molecule axis is very small, so the level spacing is large and this mode is not active at room temperature. The heat capacity at constant pressure is $C_P = C_V + Nk_B T$.

Quiz Problem 4. In modeling the earth's atmosphere using the equilibrium isothermal model, the gas pressure decreases exponentially with altitude. Explain how this is reconciled with the equilibrium condition of constant pressure found in our discussion of thermodynamics in Part 1 of the course.

Solution

The condition that the pressure of a thermodynamic system such as a gas must be a constant applies when there are no externally applied potentials in the system. An external potential such as a gravitational field or a harmonic trapping potential as occurs in atom traps leads to a force and this force is balanced by a pressure gradient in the gas. This leads to the condition of hydrostatic equilibrium like that as used in the analysis earth's atmosphere.

Quiz Problem 5. Discuss the assumptions used in the isothermal and adiabatic models of the atmosphere.

Solution

The isothermal model assumes that the atmosphere is at constant temperature which requires that thermal equilibrium has been established in all regions of the atmosphere. The adiabatic model assumes that no heat flow occurs in the gas and is justified by the argument that the thermal conductivity of air is very low. These models are clearly limiting cases with the truth being somewhere between the two. Surprisingly the adiabatic model seems to fit the behavior of the troposphere quite well.

Quiz Problem 6. What is the Curie law? Derive this law for the spin half Ising paramagnet.

Solution

The spin half Ising paramagnet has Hamiltonian,

$$H = -\mu_s h \sum_i S_i \quad (2)$$

where $S_i = \pm 1$. The partition function and magnetization are then,

$$Z_N = (2 \cosh(\beta \mu_s h))^N; \quad m = \frac{\partial(\ln(Z_N))}{\partial(\beta h)} = N \mu_s \tanh(\beta \mu_s h) \quad (3)$$

The magnetic susceptibility is then,

$$\chi = \frac{\partial m}{\partial h} = N \beta \mu_s^2 \operatorname{sech}^2(\beta \mu_s h) \approx \frac{N \mu_s^2}{k_B T} \quad (4)$$

which shows that in the limit $h \rightarrow 0$, the susceptibility behaves as C/T which is the Curie law.

Quiz Problem 7. Paramagnets and also the Ising model in one dimension exhibit a peak in their specific heat, with the low temperature behavior being an approach to zero exponentially and at high temperature an approach to zero as $1/T^2$. Why is there a peak in the specific heat in these models. What can't we use the rule $k_B T/2$ per degree of freedom to find the specific heat?

Solution

A peak in the heat capacity usually occurs in systems where there is a gap between the ground state energy and the first excited state, though this is not the only mechanism for a peak in C_V . If the gap is Δ , the the peak in the specific heat occurs at roughly $k_B T_c \approx \Delta$, though the precise location depends on the system. The peak occurs because at the temperature the first excited state begins to be populated leading to a rapid change in the internal energy with temperature and this leads to an increase in the heat capacity

Quiz Problem 8. Explain the meaning of negative temperature in the Ising paramagnetic and how it can be used to make a magnetic cooling device.

Solution Negative temperature occurs when $\partial S/\partial E$ is negative. For the Ising paramagnet this condition applies in the energy regime $E > 0$ as the peak in the many body density of states is at $E = 0$. Magnetic cooling can be produced by preparing a state in a polarized spin configuration followed by removing the field. Once the field is removed, the system becomes more disordered and in doing so absorbs heat from its surroundings. This leads to cooling, just as gas cooling is produced when a gas expands as in conventional gas cycle refrigeration.

Quiz Problem 9. What is the meaning of the terms “spontaneous symmetry breaking” (or spontaneous magnetization) and “breaking of ergodicity” in the context of the two dimensional Ising model.

Solution In a ferromagnet there are two ground states: all up spins; or all down spins. At low temperatures, these two states are separated by a large energy barrier that is impossible to overcome at zero applied field. This system is non-ergodic as phase space is separated into regions that are not connected by any trajectories. Spontaneous symmetry breaking refers to the fact that the system chooses between two states (either all up or all down) and so the up down symmetry of the system is broken without the application of a symmetry breaking field.

Quiz Problem 10. State and give a physical explanation of the behavior of the chemical potential μ and the fugacity $z = e^{\beta\mu}$ as temperature $T \rightarrow 0$, for both the non-relativistic and ultra-relativistic Bose gas.

Solution.

For the Bose gas as temperature goes to zero, the internal energy contribution dominates. As temperature goes to zero all of the particles that are added go into the ground state, so the chemical potential goes to the ground state energy. For the ideal gas case the ground state energy is zero, so the chemical potential goes to zero. The fugacity therefore goes to one. This is true for both non-relativistic and ultra-relativistic gases. In the Bose condensed phase the fugacity remains at one up to the BEC critical temperature.

Quiz Problem 11. Write down the starting expression in the derivation of the grand partition function, Ξ_B for the ideal Bose gas, for a general set of energy levels ϵ_l , with degeneracy g_l . Carry out the sums over the energy level occupancies, n_l and hence write down an expression for $\ln(\Xi_B)$.

Solution

For the case of Bose statistics the possibilities are $n_l = 0, 1, 2, \dots, \infty$ so we find

$$\Xi_B = \prod_l \left(\sum_{n_l} e^{-\beta(\epsilon_l - \mu)n_l} \right)^{g_l} = \prod_{l=1}^M \left(\frac{1}{1 - e^{-\beta(\epsilon_l - \mu)}} \right)^{g_l} = \prod_{l=1}^M \left(\frac{1}{1 - ze^{-\beta\epsilon_l}} \right)^{g_l} \quad (5)$$

where the sums are carried out by using the formula for a geometric progression. We thus find,

$$\ln(\Xi_B) = - \sum_{l=1}^M g_l \ln(1 - ze^{-\beta\epsilon_l}) \quad (6)$$

$$\langle n_l \rangle = - \frac{1}{\beta} \frac{\partial}{\partial \epsilon_l} [\ln(\Xi_B)] = \frac{ze^{-\beta\epsilon_l}}{1 - ze^{-\beta\epsilon_l}} \quad (7)$$

Quiz Problem 12. In the condensed phase, superfluids are often described in terms of a two fluid model. Based on the analysis of the ideal Bose gas, explain the physical basis of the two fluid model.

Solution

The two fluid model considers that the Bose condensed phase is a superfluid while the particles in the excited states behave as a normal fluid. The normal fluid exhibits dissipation and viscosity, while the superfluid has very low values of viscosity and other remarkable properties such as phase coherence.

Quiz Problem 13. Why is the chemical potential of photons in a box, and also acoustic phonons in a crystal, taken to be zero?

Solution.

The lowest energy state of these systems is zero so any additional photons or phonons may be placed in this state. This is explained by the fact that photons and phonons are massless bosons that can be created and destroyed, or by arguing that we in essence have a Bose condensed phase of zero energy particles so $\mu = 0$. A more subtle and ultimately the full explanation is through an understanding of the interactions with the reservoir. In the case of massive particles the reservoir contains a very large number of the same massive particles so the exchange with the reservoir is through exchange of the same type of particle. In a photon or phonon gas, the reservoir is a system of atoms where the photons or phonons may be absorbed and re-emitted as combinations of different photons or phonons. For this reason the same amount of total free energy in the phonon or photon gas may be divided amongst an arbitrary number of particles, so the chemical potential to add another particle must be zero.

Quiz Problem 14. Derive or write down the blackbody energy density spectrum in three dimensions.

Solution. The blackbody energy density spectrum follows from the equation for the energy of the photon gas in three dimensions,

$$U = 2\left(\frac{L}{2\pi}\right)^3 \int_0^\infty dk 4\pi k^2 (\hbar kc) \frac{e^{-\beta\hbar kc}}{1 - e^{-\beta\hbar kc}} = 2\left(\frac{L}{2\pi c}\right)^3 \int_0^\infty d\omega 4\pi\omega^2 (\hbar\omega) \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \quad (8)$$

where we used $\omega = kc$. we then write,

$$\frac{U}{V} = \int d\omega u(\omega), \quad \text{where,} \quad u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Quiz Problem 15. Write down and explain the relationship between the intensity of radiation emitted by a blackbody (Stefan-Boltzmann law) and the energy density of a photon gas in the blackbody.

Solution. The relationship between the intensity and the energy density of blackbody radiation is,

$$I = \frac{c}{4} \frac{U}{V} = \sigma T^4 \quad (10)$$

The factor $c/4$ is explained as follows: The factor of c converts the energy density of an EM wave into the intensity of radiation crossing a surface whose surface normal is in the same direction as the direction of wave propagation. The factor of $1/4$ has two pieces. First we note that emission from the surface of a blackbody is isotropic so half of the radiation is emitted back into the blackbody. Moreover, the amount of radiation emitted to the exterior is also in all directions on a hemisphere. To find the radiation emitted in the normal direction, we take the component of the electric field in the normal direction, leading to a factor of $\cos(\theta)$. However the intensity is the square of the electric field, so it comes with a factor of $\cos^2(\theta)$. The average of $\cos^2(\theta)$ leads to the second factor of $1/2$.

Quiz Problem 16. Explain the physical origin of the cosmic microwave background (CMB) blackbody spectrum of the universe. It is currently at a temperature of $T_{CMB} = 2.713K$. If the universe is expanding at a constant rate $L(t) = H_0 t$, where H_0 is a constant what is the expected behavior of the temperature $T_{CMB}(t)$.

Solution. During the “photon epoqe” of the early universe that is believed to have existed during the period from 10 seconds after the big bang to 377 thousand years after the big bang (that is believed to have occurred roughly 13.7 billions years ago), the universe consisted of a gas of charged particles and photons that was equilibrated. At around 380 thousand years after the big bang, Hydrogen and Helium began to form, reducing the scattering of photons and the universe became “transparent”. The cosmic microwave background is a remnant of the photon gas that existed 380 thousand years ago. Assuming that the photon gas making up the CMB has not changed significantly due to scattering since that time, we can relate the temperature of the CMB to the size of the universe by assuming that the energy in the photon gas is conserved, so that,

$$U = \text{constant} = L(t)^3 \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4 \quad (11)$$

where $L(t)$ is the size of the universe. We then find $T \propto L^{-3/4}$.

Quiz Problem 17. Derive or write down the spectral energy density for blackbody radiation in a universe with two spatial dimensions.

Solution. The blackbody energy density spectrum follows from the equation for the energy of the photon gas in two dimensions,

$$U = 2\left(\frac{L}{2\pi c}\right)^2 \int_0^\infty d\omega \, 2\pi\omega(\hbar\omega) \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = L^2 \int d\omega \, u(\omega) \quad (12)$$

Note that here I kept the two polarizations of light even though one of them is along the third direction. We then have,

$$u(\omega) = \frac{\hbar}{\pi c^2} \frac{\omega^2}{e^{\beta\hbar\omega} - 1}$$

Quiz Problem 18. Derive or write down the Debye theory for the internal energy for phonons in a square lattice. Derive the low and high temperature limits of the internal energy and specific heat for this system.

Solution. The energy density for the Debye model for the case of a square lattice comes from assuming that phonons are an ideal Bose gas where there is one acoustic mode per atom. The chemical potential is taken to be zero, so we have,

$$U = 2\left(\frac{L}{2\pi}\right)^2 \int_0^{k_D} dk \, 2\pi k(\hbar k v_s) \frac{e^{-\beta\hbar k v_s}}{1 - e^{-\beta\hbar k v_s}} \quad (14)$$

where,

$$N_A = \left(\frac{L}{2\pi}\right)^2 \int_0^{k_D} 2\pi k \, dk, \quad \text{so that} \quad k_D = \left(\frac{4\pi N_A}{L^2}\right)^{1/2} \quad (15)$$

The factor of two in the front of the energy equation takes into account the fact that there are two phonon modes for the square lattice. This is a rough approximation as only one of the the two modes has the dispersion relation $\epsilon_p = p v_s$.

We define $x = \beta p v_s$, and $k_B T_D = \hbar k_D c$ leading to,

$$\frac{U}{L^2} = 4T \left(\frac{T}{T_D}\right)^2 \int_0^{T_D/T} dx \frac{x^2}{e^x - 1} \quad (16)$$

In the low temperature and high temperature limits this reduces to,

$$\frac{C_V}{N k_B} \sim \left(\frac{T}{T_D}\right)^2, \quad T \ll T_D; \quad C_V = 2N k_B \quad T \gg T_D \quad (17)$$

Quiz Problem 19. On a graph, illustrate the behavior of the chemical potential μ and the fugacity $z = e^{\beta\mu}$ of Bose, Fermi and classical gases as a function of temperature. What is the chemical potential and fugacity of the the Fermi and Bose gases as temperature $T \rightarrow \infty$ and as $T \rightarrow 0$.

Solution

At low temperatures the chemical potential of quantum gases approaches the mean of the energy of the highest occupied and lowest unoccupied energy levels. For Fermions with a negligible gap at the Fermi energy the chemical potential at $T = 0$ is the same as the Fermi energy. For Bosons, all particles may be in the lowest energy level and for gases this is close to zero for large volumes, so the chemical potential of the Bose gas is zero at $T = 0$, $V \infty$. At high temperatures the Bose and Fermi gases act like a classical gas so their chemical potential is given by, $\mu = k_B T \ln(N\lambda^3/V)$ to leading order. The fugacity of all three gases approaches zero at sufficiently high T , while at low T the fugacity of the Bose gas approaches one while that of the Fermi gas diverges.

Quiz Problem 20. Write down the starting expression in the derivation of the grand partition function, Ξ_F for the ideal Fermi gas, for a general set of energy levels ϵ_l . Carry out the sums over the energy level occupancies, n_l and hence write down an expression for $\ln(\Xi_F)$.

Solution

$$\Xi_F = \sum_{n_1} \dots \sum_{n_M} e^{-\beta \sum_{l=1}^M (\epsilon_l - \mu) n_l} = \prod_{l=1}^M (1 + e^{-\beta(\epsilon_l - \mu)}) = \prod_{l=1}^M (1 + ze^{-\beta\epsilon_l}) \quad (18)$$

where $z = e^{\beta\mu}$ and each sum is over the possibilities $n_l = 0, 1$ as required for Fermi statistics. We thus find,

$$\ln(\Xi_F) = \sum_{l=1}^M \ln(1 + ze^{-\beta\epsilon_l}) \quad (19)$$

Quiz Problem 21. White dwarf stars are stable due to electron degeneracy pressure. Explain the physical origin of this pressure.

Solution

Even in the ground state, the internal energy of the Fermi gas is positive. This is due to the fact that only one Fermion can be in each energy level so high energy states are occupied at zero temperature. As the density increases, the Fermi energy or energy of the highest occupied state, increases. The pressure is the rate of change of the energy with volume so the pressure increases with the density. This “degeneracy pressure” opposes gravitational collapse and stabilizes white dwarf stars.

Quiz Problem 15. Explain the physical origins of the paramagnetic and diamagnetic contributions to the magnetization of the free electron gas.

Solution. The paramagnetic contribution to the magnetization of the free electron gas is the change in the spin polarization due to the application of a magnetic field. The diamagnetic contribution to the magnetization is due to changes in the electron orbitals due to the application of a magnetic field. The diamagnetic contribution can occur even if there is no net spin. To a first approximation, we can add the paramagnetic and diamagnetic contributions. When a paramagnetic contribution occurs, these two contributions are usually of opposite sign.

Assigned problems

Assigned Problem 1. From the density of states for an ideal monatomic gas $\Omega(E)$ given in Eq. (26) of the notes, find the Sackur-Tetrode equation for the entropy, Eq. (27) of the notes.

Solution.

$$\Omega(E) = \frac{2\pi^{1/2} V^N (2\pi m E)^{3N/2-1/2}}{N! h^{3N} (\frac{3N}{2} - 1)!} \quad (20)$$

Using Stirling’s approximation and dropping constants, we have

$$k_B \ln(\Omega(E)) = k_B N \left[\ln\left(\frac{V}{h^3}\right) - \ln(N) + 1 - \frac{3}{2} \ln\left(\frac{3N}{2}\right) + \frac{3}{2} + \frac{3}{2} \ln(2\pi m U) \right] = N k_B \left[\ln\left[\frac{V}{N} \left(\frac{4\pi m U}{3N h^2}\right)^{3/2}\right] + \frac{5}{2} \right] \quad (21)$$

Assigned Problem 2. Using the canonical partition function for the ideal gas, show that,

$$(\delta E)^2 = k_B T^2 C_v \quad (22)$$

Solution.

From Part 1 page 28, and using the ideal classical gas expression $Z_N = V^N / (N! \lambda^{3N})$, we have,

$$\delta E^2 = \frac{\partial^2 \ln(Z)}{\partial \beta^2} = \delta E^2 = \frac{\partial^2}{\partial \beta^2} \left[\ln \left(\frac{V^N (2\pi m)^{3N/2}}{N! h^{3N}} - \frac{3N}{2} \ln(\beta) \right) \right] = \frac{3}{2} N (k_B T)^2 = k_B T^2 C_V \quad (23)$$

where $C_V = 3Nk_B/2$ for the classical monatomic non-relativistic ideal gas in three dimensions.

Assigned Problem 3. Using the grand partition function of the ideal classical gas show that,

$$(\delta N)^2 = N k_B T \rho \kappa_T \quad (24)$$

Solution. From Part 1 page 28, and using the expression for the grand partition function for the classical gas we have,

$$(\delta N)^2 = (k_B T)^2 \frac{\partial^2 \ln(\Xi)}{\partial \mu^2} = (k_B T)^2 \frac{V}{\lambda^3 \beta^2} e^{\beta \mu} = \frac{V}{\lambda^3} e^{\beta \mu} = \frac{PV}{k_B T} = N \quad (25)$$

where we used (II.25) to write $\alpha z = \ln(\Xi) = PV/k_B T$. Also the right hand side of Eq. (17) for the classical ideal gas is,

$$N k_B T \rho \kappa_T = k_B T \frac{N^2}{V} \kappa_T = k_B T \frac{N^2}{PV} = N \quad (26)$$

where we used Eq. (II.17) for κ_T for the ideal gas

Assigned Problem 4

Consider a gas of N atoms in volume V . Each atom has one unpaired electron in its outer energy level that is a zero angular momentum state, i.e. the ground state is $l = 0$ and has an unpaired electron. A magnetic field is applied to this gas. Write down the canonical and grand canonical partition functions of this system taking into account center of mass and spin degrees of freedom, assuming that the magnetic interaction is paramagnetic with no spin-orbit coupling.

Solution The translational and spin degrees of freedom do not interact, so the canonical partition function is simply a product of the two subsystem canonical partition functions,

$$Z_N = \frac{V^N}{N! \lambda^{3N}} (2 \cosh(\beta \mu_s h))^N \quad (27)$$

The grand partition function is,

$$\Xi = \sum_N z^N Z_N = e^{\alpha z} \quad \text{where} \quad \alpha = \frac{V}{\lambda^3} 2 \cosh(\beta \mu_s h) \quad (28)$$

Assigned Problem 5.

Consider a gas where each atom can have one of two energy levels ϵ_0 and ϵ_1 . Find the average energy per particle taking into account the ideal monatomic gas behavior and the two internal energy levels. Find the specific heat of the system in the limits $T \rightarrow 0$ and $T \rightarrow \infty$.

Solution The canonical partition function is again a product of the two subsystem partitions functions so that,

$$Z_N = \frac{V^N}{N! \lambda^{3N}} (e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1})^N \quad (29)$$

so the internal energy is,

$$U = U_n + U_i = \frac{3}{2} N k_B T + N \frac{\epsilon_0 e^{-\beta\epsilon_0} + \epsilon_1 e^{-\beta\epsilon_1}}{e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}} \quad (30)$$

where U_n is the contribution from the nuclear or translational degrees of freedom while U_i is the contribution from the internal degrees of freedom.

Assigned Problem 6.

Consider a dissociating system at thermal equilibrium, where,



Ignore the internal degrees of freedom of both species and take the mass of an A particle to be twice that of a B particle. Given that $N = 2N_A + N_B$ is fixed, that the volume is fixed, and that the binding energy of A is ϵ , find the temperature dependence of N_A and N_B and find a simple expression for the ratio $\langle N_B \rangle^2 / \langle N_A \rangle$.

Solution

In this system there is an exchange of particles between the subsystem consisting of B atoms and the subsystem A consisting of diatomic molecules with 2 B atoms. If there are N_A molecules and N_B free atoms the canonical partition functions for the two subsystems is,

$$Z_A = \frac{V^{N_A}}{N_A! \lambda_A^{3N_A}} e^{\beta\epsilon N_A}; \quad Z_B = \frac{V^{N_B}}{N_B! \lambda_B^{3N_B}} \quad (32)$$

where ϵ is the binding energy of the molecule. The equilibrium condition is that the chemical potential of an atom of B must be the same whether it is bound in a molecule or free. The chemical potential of a free B atom is,

$$\mu_B = \frac{\partial}{\partial N_B} (-k_B T \ln Z_B) = -k_B T \ln \left(\frac{V}{N_B \lambda_B^3} \right) \quad (33)$$

while the chemical potential of an A dimer consisting of two bound B atoms is,

$$\mu_A = \frac{\partial}{\partial N_A} (-k_B T \ln Z_A) = -k_B T \ln \left(\frac{V}{N_A \lambda_A^3} \right) - \epsilon \quad (34)$$

At equilibrium a B atom has the same chemical potential whether it is free or bound into an A dimer. We then have the condition,

$$\mu_B = \frac{1}{2} \mu_A \quad (35)$$

and using this condition we find the relation,

$$\frac{N_B^2}{N_A} = \frac{V}{8\lambda_A^3} e^{-\beta\epsilon} \quad (36)$$

where we used $m_A = 2m_B$ so that $\lambda_B^2 = 2\lambda_A^2$. Using this relation in combination with the condition $N = 2N_A + N_B$, we can find the behavior of N_B as a function of temperature by solving,

$$\frac{2N_B^2}{N - N_B} = \frac{V}{8\lambda_A^3} e^{-\beta\epsilon} = x \quad (37)$$

so that,

$$2N_B^2 + xN_B - xN = 0; \quad \text{so} \quad N_B = \frac{1}{4} [-x \pm \sqrt{x^2 + 8Nx}] = \frac{x}{4} [-1 \pm \sqrt{1 + 8N/x}] \quad (38)$$

The grand partition function may also be used to find the ratio $\langle N_B \rangle^2 / \langle N_A \rangle$, by realizing that $\Xi = \Xi_A \Xi_B$ and using,

$$\Xi_B = \sum_{N_B} z_B^{N_B} Z_B(N) = \text{Exp}[z_B V / \lambda_B^3]; \quad \Xi_A = \sum_{N_A} z_A^{N_A} Z_A(N) = \text{Exp}[z_A V / \lambda_A^3] \quad (39)$$

The average number of atoms of each type is given by,

$$\langle N_B \rangle = \frac{\partial \ln(\Xi_B)}{\partial(\beta\mu_B)} = e^{\beta\mu_B} \frac{V}{\lambda_B^3} \quad (40)$$

and

$$\langle N_A \rangle = \frac{\partial \ln(\Xi_A)}{\partial(\beta\mu_A)} = e^{\beta\mu_A} e^{\beta\epsilon} \frac{V}{\lambda_A^3} \quad (41)$$

Now use the fact that $\mu_A = 2\mu_B$ and take the ratio $\langle N_B \rangle^2 / \langle N_A \rangle$ to reproduce the result found earlier.

Assigned Problem 7.

Calculate the specific heat of the model defined by the Hamiltonian Eq. (51) of the text. Is there a peak in the specific heat? Why or why not?

Solution The internal energy per spin of this model is $-hm$ so that,

$$U = -\mu_s h N \left[\coth(\beta\mu_s h) - \frac{1}{\beta\mu_s h} \right] \quad (42)$$

Defining $x = \beta\mu_s h$, we find the specific heat to be

$$C_h = \mu_s h k_B N \left[\frac{1}{x^2} - \frac{1}{\sinh^2(x)} \right] \quad (43)$$

This increases monotonically with temperature and has no peak.

Assigned Problem 8.

Generalize the transfer matrix method for the one dimensional Ising model to the case of an applied field (i.e. combine the Hamiltonians in Eq. (41) and (55)) and calculate the magnetic susceptibility at low field. Does it obey a Curie law?

Solution It is convenient to write the Hamiltonian in the form,

$$H = -J \sum_i S_i S_{i+1} - \mu_s h \sum_i \frac{1}{2} (S_i + S_{i+1}) \quad (44)$$

as this makes the transfer matrix more symmetric. We then find,

$$T_{++} = e^{\beta J + \beta\mu_s h}; \quad T_{+-} = e^{-\beta J} = T_{-+}; \quad T_{--} = e^{\beta J - \beta\mu_s h} \quad (45)$$

and we seek the largest eigenvalue of this matrix. Defining $\alpha = e^{\beta J}$; $\gamma = e^{\beta\mu_s h}$ the characteristic equation for \hat{T} is,

$$\lambda^2 - \alpha \left(\gamma + \frac{1}{\gamma} \right) + \alpha^2 - \frac{1}{\alpha^2} = \lambda^2 - 2e^{\beta J} \cosh(\beta\mu_s h) \lambda + 2\sinh(2\beta J) \quad (46)$$

which has solutions,

$$\lambda_{\pm} = \frac{1}{2} \left[2e^{\beta J} \cosh(\beta\mu_s h) \pm \sqrt{4e^{2\beta J} \cosh^2(\beta\mu_s h) - 8\sinh(2\beta J)} \right] \quad (47)$$

which simplifies to,

$$\lambda_{\pm} = e^{\beta J} [\cosh(\beta\mu_s h) \pm \sqrt{\sinh^2(\beta\mu_s h) + e^{-4\beta J}}] \quad (48)$$

The eigenvalue with the plus sign dominates, so we find the magnetization using,

$$m = \frac{M}{N} = \frac{1}{N} \frac{\partial}{\partial(\beta h)} \ln(Z) = \frac{\partial}{\partial(\beta h)} \ln \left(e^{\beta J} [\cosh(\beta\mu_s h) + \sqrt{\sinh^2(\beta\mu_s h) + e^{-4\beta J}}] \right) \quad (49)$$

This simplifies to,

$$m(h, T) = \mu_s \frac{\text{Sinh}(\beta h \mu_s)}{[\text{Sinh}^2(\beta \mu_s h) + e^{-4\beta J}]^{1/2}} \rightarrow \beta \mu_s^2 h e^{2\beta J}; \quad 1d \text{ Ising}. \quad (50)$$

where the latter form is correct in the small h limit. The magnetic susceptibility at small field is then

$$\chi = \frac{\partial m}{\partial h} \rightarrow \frac{\mu_s^2}{k_B T} e^{2\beta J} \quad \text{as } h \rightarrow 0 \quad (51)$$

This diverges exponentially at low T due to the incipient phase transition in the 1D Ising model.

Assigned Problem 9.

Consider the Peierls argument for the two dimensional spin half Ising model on a square lattice, where the topological excitation is a flat domain wall dividing a domain of up spins and down spins. Does this argument suggest that finite temperature spontaneous symmetry breaking is possible in the two dimensional Ising model?

Solution

In the two dimensional Ising model the Peierls estimate of the difference in free energy between the ground state and a state with one domain wall in it is,

$$\delta F = 2JL - k_B T \ln(L). \quad (52)$$

Clearly the energy cost is larger than the entropy gain so the ordered state is stable.

Assigned Problem 10. Using the energy levels of a particle in a box and MB statistics to fill the energy levels, derive the classical gas thermodynamics.

Solution

The grand partition function is given by,

$$\Xi_{MB} = \prod_i \text{Exp}[g_i e^{-\beta(\epsilon_i - \mu)}] \quad \text{so} \quad \ln(\Xi_{MB}) = \sum_i g_i e^{-\beta(\epsilon_i - \mu)} \quad (53)$$

Using the energy levels of a particle in a box and taking the continuum limit and setting $g_i = 1$, this becomes,

$$\ln(\Xi_{MB}) = \left(\frac{L}{2\pi}\right)^3 \int_0^\infty 4\pi k^2 z e^{-\beta p^2/(2m)} \quad (54)$$

Using cartesian co-ordinates we find,

$$\ln(\Xi_{MB}) = z \left(\frac{L}{h}\right)^3 \left(\int_{-\infty}^\infty e^{-\beta p^2/(2m)}\right)^3 = zV \left(\frac{2m\pi}{\beta}\right)^{3/2} \quad (55)$$

so that,

$$\ln(\Xi_{MB}) = z \frac{V}{\lambda^3} \quad (56)$$

as for the classical gas.

Assigned Problem 11. In lectures we showed that at high temperatures the equation of state of the Bose gas reduces to the ideal classical gases. Derive the next term in the expansion of the equation of state of the ideal Bose at high temperatures.

Solution. For the Bose case, expanding to second order gives

$$\frac{N\lambda^3}{V} = g_{3/2}(z) = z + \frac{z^2}{2\sqrt{2}} \approx z_1 + \frac{z_0^2}{2\sqrt{2}} \quad (57)$$

where $z_0 = N\lambda^3/V$ is the leading order solution, we then find,

$$z_1 = z_0 - \frac{z_0^2}{2\sqrt{2}} \quad (58)$$

We substitute this into the second order expansion of the equation of state,

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} \left[z + \frac{z^2}{4\sqrt{2}} + \dots \right] \approx \frac{1}{\lambda^3} \left[z_1 + \frac{z_1^2}{4\sqrt{2}} + \dots \right] \approx \frac{1}{\lambda^3} \left[z_0 - \frac{z_0^2}{2\sqrt{2}} + \left(\frac{z_0 - \frac{z_0^2}{2\sqrt{2}}}{4\sqrt{2}} \right)^2 + \dots \right] \quad (59)$$

keeping terms to order z_0^2 gives,

$$\frac{P}{k_B T} = \frac{z_0}{\lambda^3} \left[1 - z_0 \left(\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} \right) + \dots \right] \quad (60)$$

substituting $z_0 = N\lambda^3/V$ gives,

$$\frac{PV}{Nk_B T} = 1 - \frac{1}{4\sqrt{2}} \frac{\lambda^3 N}{V} + \dots \quad \text{Bose gas} \quad (61)$$

The pressure is lower than the classical gas due to the effective attraction of Bosons.

Assigned Problem 12. Derive expressions for Ξ , PV , N/V and U for the non-relativistic Bose gas in one and two dimensions. Write general expressions that are valid in any dimension. Find the leading order terms in the high temperature expansions for these quantities. Discuss the behavior of the non-relativistic Bose gas at low temperatures in one and two dimensions. Is a finite temperature Bose condensation predicted? Explain your reasoning.

Solution. For a Bose gas with dispersion relation $\epsilon_p = p^2/2m$ in d dimensions

$$\frac{P}{k_B T} = \frac{1}{\lambda^d} g_{d/2+1}(z) - \frac{1}{L^d} \ln(1-z); \quad \frac{N}{L^d} = \frac{1}{\lambda^d} g_{d/2}(z) + \frac{1}{L^d} \frac{z}{1-z}; \quad \frac{U}{L^d} = \frac{d}{2} \frac{k_B T}{\lambda^d} g_{d/2+1}(z) \quad (62)$$

These results are found from the integral forms in one and two dimensions below along with the three dimensional result derived in lectures.

$$\ln(\Xi_B) = -\frac{L}{2\pi\hbar} \int_0^\infty 2dp \ln(1 - ze^{-\beta p^2/2m}) - \ln(1-z); \quad 1 - \text{dimension} \quad (63)$$

and

$$\ln(\Xi_B) = -\left(\frac{L}{2\pi\hbar}\right)^2 \int_0^\infty 2\pi p dp \ln(1 - ze^{-\beta p^2/2m}) - \ln(1-z) \quad 2 - \text{dimension} \quad (64)$$

A series expansion as carried out for these integrals and the similar forms for N/V and U lead to the results above. To leading order in the fugacity of the equation for N/V , we find the chemical potential to be the same as that of the ideal classical gas in d dimensions, i.e. $z = N(\lambda/L)^d$. The dimensional dependence comes from the different powers of the factor $(L/h)^d$, and the factors of p in the integral. The integrals that are needed are,

$$\int_0^\infty x e^{-x^2 l} dx = \frac{1}{2l}; \quad \int_0^\infty e^{-x^2 l} dx = \frac{1}{2} \left(\frac{\pi}{l}\right)^{1/2} \quad (65)$$

We then have,

$$\frac{PL}{k_B T} = \ln(\Xi_B) = \frac{L}{2\pi\hbar} \left(\frac{2m}{\beta}\right)^{1/2} \sum_{l=1}^{\infty} \frac{z^l}{l} \int_0^{\infty} 2dx e^{-x^2 l} - \ln(1-z); \quad 1 - \text{dimension} \quad (66)$$

and

$$\frac{PL^2}{k_B T} = \ln(\Xi_B) = \left(\frac{L}{2\pi\hbar}\right)^2 \left(\frac{2m}{\beta}\right) \sum_{l=1}^{\infty} \frac{z^l}{l} \int_0^{\infty} 2\pi x dx e^{-x^2 l} - \ln(1-z); \quad 2 - \text{dimensions} \quad (67)$$

which reduce to the expression given above for $P/(k_B T)$.

At any temperature, the chemical potential of the ideal non-relativistic Bose gas in dimensions less than $2 + \delta$ cannot be one in the thermodynamic limit, as,

$$g_n(z) = \sum_l \frac{z^l}{l^n} \quad (68)$$

diverges for $z = 1$ and $n \leq 1$. Since z cannot approach one, the term $z/(V(1-z))$ approaches zero in the thermodynamic limit, indicating that it is impossible for a finite fraction of the particles to be in the ground state. There is therefore no BEC phase transition at finite temperature in one and two dimensional ideal non-relativistic Bose gases.

Assigned Problem 13. For the 3-D non-relativistic case, find the entropy of the ideal Bose gas in the condensed phase $T < T_c$. Compare to the classical gas.

Solution. Using the thermodynamic relation, $U = TS - PV + \mu N$, we find,

$$TS = U + PV - \mu N = \frac{5}{2}PV - \mu N \quad (69)$$

For the Bose gas at $T < T_c$ where $\mu = 0$, we have $P/k_B T = \zeta(5/2)/\lambda^3$ and using $PV = 2U/3$ we find,

$$TS = \frac{5}{2}PV = \frac{5}{2} \frac{k_B T V}{\lambda^3} \zeta(5/2) \quad (70)$$

The entropy of the Bose gas in the condensed phase is thus proportional to $T^{3/2}$, which goes to zero at low temperature. The entropy of the classical gas is given by Eq. (16) of lecture notes part 2, so that,

$$S = Nk_B \left[\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2} \right] \quad (71)$$

The classical gas has higher entropy than the Bose gas in the condensed phase. In the condensed phase the number of excited state particles grows at $T^{3/2}$ and only these particles contribute to the entropy of the Bose gas. The ground state particles do not contribute to the entropy and as the temperature is reduced almost all particles are in the ground state.

Assigned Problem 14. Show that a d -dimensional Bose gas with dispersion relation $\epsilon_p = cp^s$ obeys the relation,

$$P = \frac{s}{d} \frac{U}{L^d} \quad (72)$$

Solution. The equations for a Bose gas with this dispersion relation in d dimensions is written as,

$$\frac{P}{k_B T} = -\frac{c_d}{h^d} \int_0^{\infty} dp p^{d-1} \ln(1 - ze^{-\beta cp^s}); \quad \frac{U}{L^d} = \frac{c_d}{h^d} \int_0^{\infty} dp p^{d-1} (cp^s) \frac{ze^{-\beta cp^s}}{1 - ze^{-\beta cp^s}} \quad (73)$$

Integrating the pressure equation by parts gives,

$$I_1 = -\int_0^{\infty} dp p^{d-1} \ln(1 - ze^{-\beta cp^s}) = -\frac{1}{d} p^d \ln(1 - ze^{-\beta cp^s}) \Big|_0^{\infty} + \frac{s\beta}{d} \int_0^{\infty} dp p^{d-1} (cp^s) \frac{ze^{-\beta cp^s}}{1 - ze^{-\beta cp^s}} \quad (74)$$

The first term on the right hand side is zero and comparison of the remaining integral with the energy equation proves relation (63).

Assigned Problem 15. Find the thermodynamic properties, PV , U , S , C_V , N of a photon gas in d dimensions. Show that the entropy per photon is independent of temperature.

Solution. We use the relations,

$$N = 2 \left(\frac{L}{2\pi\hbar} \right)^d \int_0^\infty c_d p^{d-1} dp \frac{e^{-\beta pc}}{1 - e^{-\beta pc}} = 2c_d \left(\frac{L}{h} \right)^d \left(\frac{1}{\beta c} \right)^d \int_0^\infty dx \frac{x^{d-1}}{e^x - 1} \quad (75)$$

and

$$U = 2 \left(\frac{L}{2\pi\hbar} \right)^d \int_0^\infty c_d p^{d-1} dp (pc) \frac{e^{-\beta pc}}{1 - e^{-\beta pc}} = 2c_d c \left(\frac{L}{h} \right)^d \left(\frac{1}{\beta c} \right)^{d+1} \int_0^\infty dx \frac{x^d}{e^x - 1} \quad (76)$$

along with $PV = sU/d$, with $s = 1$ and the integral,

$$\int_0^\infty \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad (77)$$

to find,

$$U = 2c_d c d! \zeta(d+1) \left(\frac{L}{h} \right)^d \left(\frac{k_B}{c} \right)^{d+1} T^{d+1}; \quad N = 2c_d (d-1)! \zeta(d) \left(\frac{L}{h} \right)^d \left(\frac{k_B}{c} \right)^d T^d \quad (78)$$

We also have,

$$TS = U + PV - \mu N = (d+1)U/d \propto T^{d+1} \quad (79)$$

From this it is evident that both S and N are proportional to T^d , so S/N is temperature independent. Also,

$$C_V = \frac{\partial U}{\partial T} = 2(d+1)! \zeta(d+1) c_d c \left(\frac{L}{h} \right)^d \left(\frac{k_B}{c} \right)^{d+1} T^d \quad (80)$$

Assigned Problem 16. Find the thermodynamic properties, U and C_V for the Debye phonon model in d dimensions.

Solution. We use the relation,

$$U = d \left(\frac{L}{2\pi\hbar} \right)^d \int_0^{p_D} c_d p^{d-1} dp (pv_s) \frac{e^{-\beta pv_s}}{1 - e^{-\beta pv_s}} = dc_d v_s \left(\frac{L}{h} \right)^d \left(\frac{k_B}{v_s} \right)^{d+1} T^{d+1} \int_0^{x_D} dx \frac{x^d}{e^x - 1} \quad (81)$$

where the factor of d in front ensures that we recover the high temperature equipartition result. $x_D = \beta p_D v_s$ and we define the Debye temperature through $k_B T_D = p_D v_s$, so that $x_D = \beta k_B T_D = T_D/T$. We also use the integral,

$$\int_0^\infty \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad (82)$$

to find that at low temperatures $T_D/T \rightarrow \infty$, so the behavior is like that of the photon gas in d dimensions, with $c \rightarrow v_s$, and multiplied by $d/2$ due to the difference in degeneracy (2 for photons, d for phonons). We then have ,

$$U = dc_d v_s d! \zeta(d+1) \left(\frac{L}{h} \right)^d \left(\frac{k_B}{v_s} \right)^{d+1} T^{d+1}; \quad C_V = \frac{\partial U}{\partial T} = d(d+1)! \zeta(d+1) c_d v_s \left(\frac{L}{h} \right)^d \left(\frac{k_B}{v_s} \right)^{d+1} T^d \quad (83)$$

At high temperatures, the behavior is like that of a classical gas in a harmonic potential so that $U = dNk_B T$, $C_V = Nd k_B$, and $PV = 2U/d$.

Assigned Problem 17. Carry through the analysis of the Bose gas for the case of massive ultrarelativistic particles where $\mu \neq 0$. What is the lower critical dimension for a BEC transition in this case?

Solution. For the ultrarelativistic Bose gas, we use, $\epsilon_k = \hbar kc$, so that,

$$N = \left(\frac{L}{2\pi\hbar} \right)^3 \int_0^\infty 4\pi k^2 dk \frac{ze^{-\beta\hbar kc}}{1 - ze^{-\beta\hbar kc}} + \frac{z}{1-z} \quad (84)$$

which reduces to,

$$N = \frac{V}{2\pi^2} \frac{1}{(\beta\hbar c)^3} \int_0^\infty dx x^2 z e^{-x} \sum_{l=0}^\infty z^l e^{-xl} + \frac{z}{1-z}. \quad (85)$$

where we used the expansion $1/(1-x) = 1 + x + x^2 + x^3 \dots$. Using the integral,

$$\int_0^\infty dx x^{s-1} e^{-lx} = \frac{1}{l^s} \int_0^\infty y^{s-1} e^{-y} dy = \frac{(s-1)!}{l^s} \rightarrow (s=3) \quad \frac{2}{l^3} \quad (86)$$

$$N = \frac{V}{2\pi^2} \frac{1}{(\beta\hbar c)^3} \int_0^\infty dx \sum_{l=1}^\infty \frac{2z^l}{l^3} + \frac{z}{1-z}. \quad (87)$$

Since for $z = 1$, the series is convergent, there is the possibility of Bose condensation at finite temperature and by setting $z = 1$ we find the critical condition,

$$N = \frac{V}{\pi^2} \frac{1}{(\beta\hbar c)^3} \zeta(3) \quad (88)$$

A similar calculation shows that in two dimensions the ultrarelativistic ideal Bose gas has a phase transition at finite temperature while in one dimension a finite temperature BEC transition does not occur.

Assigned Problem 18. At high temperatures we found that the ideal quantum gases reduce to the ideal classical gases. Derive the next term in the expansion of the equation of state of the ideal Fermi gas at high temperatures, and verify that,

$$\frac{PV}{Nk_B T} = 1 + \frac{1}{4\sqrt{2}} \frac{\lambda^3 N}{V} + \dots \quad \text{Fermi gas} \quad (89)$$

The pressure of the ideal Fermi gas is higher than that of the classical gas at the same temperature and volume. Why? Carry out a similar expansion for the Bose gas. Is the pressure higher or lower than the ideal classical gas at the same values of T, V ? Why?

Solution. For the Bose case, expanding to second order gives

$$\frac{N\lambda^3}{V} = g_{3/2}(z) = z + \frac{z^2}{2\sqrt{2}} \approx z_1 + \frac{z_0^2}{2\sqrt{2}} \quad (90)$$

where $z_0 = N\lambda^3/V$ is the leading order solution, we then find,

$$z_1 = z_0 - \frac{z_0^2}{2\sqrt{2}} \quad (91)$$

We substitute this into the second order expansion of the equation of state,

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} \left[z + \frac{z^2}{4\sqrt{2}} + \dots \right] \approx \frac{1}{\lambda^3} \left[z_1 + \frac{z_1^2}{4\sqrt{2}} + \dots \right] \approx \frac{1}{\lambda^3} \left[z_0 - \frac{z_0^2}{2\sqrt{2}} + \left(\frac{z_0 - \frac{z_0^2}{2\sqrt{2}}}{4\sqrt{2}} \right)^2 + \dots \right] \quad (92)$$

keeping terms to order z_0^2 gives,

$$\frac{P}{k_B T} = \frac{z_0}{\lambda^3} \left[1 - z_0 \left(\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} \right) + \dots \right] \quad (93)$$

substituting $z_0 = N\lambda^3/V$ gives,

$$\frac{PV}{Nk_B T} = 1 - \frac{1}{4\sqrt{2}} \frac{\lambda^3 N}{V} + \dots \quad \text{Bose gas} \quad (94)$$

Analysis for the Fermi gas is the same, except that the sign on the correction term is positive. The Bose gas has reduced pressure as compared to the classical gas at the same temperature, while the Fermi gas has higher pressure than the classical case. This expansion can be extended to higher order and in general is written as,

$$\frac{PV}{Nk_B T} = \sum_{l=1}^{\infty} a_l \alpha^{l-1}; \quad \text{where} \quad \alpha = \frac{\lambda^3 N}{V} \quad (95)$$

This expansion is valid when α is small, which means low density and/or high temperatures. In general expansions of this type are called **virial expansions** and have played an important role in characterizing interactions in gases. In classical gases the second virial coefficient a_2 is determined by the strength of the pair interactions, as we shall see in Part 3 of the course. Here the terms $l > 1$ are due to quantum effects. In real quantum gases, both quantum effects and interactions can be important. Recall that our condition for quantum effects to be important was that the interparticle spacing $L_c < \lambda$. When this is true α is significant and more terms are required in the virial expansion.

Assigned Problem 19. By expanding the denominator of the integral, $1/(1+y)$ for small $y = ze^{-x^2}$ show that,

$$f_{3/2}(z) = \frac{4}{\pi^{1/2}} \int_0^{\infty} dx x^2 \frac{ze^{-x^2}}{1+ze^{-x^2}} = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{3/2}} \quad (96)$$

Solution.

We use the expansion

$$\frac{1}{1+y} = (1 - y + y^2 - y^3 \dots); \quad \text{and} \quad \int_0^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}} \quad (97)$$

so that,

$$f_{3/2}(z) = \frac{4}{\pi^{1/2}} \int_0^{\infty} dx x^2 \frac{ze^{-x^2}}{1+ze^{-x^2}} = \sum_{l=1}^{\infty} (-1)^{l+1} z^l \int_0^{\infty} dx x^2 e^{-lx^2} = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{3/2}} \quad (98)$$

Assigned Problem 20a. Derive expressions for Z , Ξ , PV , μ and U for the classical gas in one and two dimensions. Are the results what you expect? How do they compare with the result in three dimensions. Write general expressions that are valid in any dimension.

Solution In d -dimensions, the partition functions are,

$$Z = \frac{L^{dN}}{\lambda^{dN} N!}, \quad \Xi = e^{\alpha z} \quad \alpha = \left(\frac{L}{\lambda} \right)^d \quad (99)$$

$$F = -k_B T \ln(Z) = -k_B T \ln \left(\frac{L^{dN}}{\lambda^{dN} N!} \right) \quad (100)$$

$$S = - \left(\frac{\partial F}{\partial T} \right)_{L^d, N} = k_B \ln \left(\frac{L^{dN}}{\lambda^{dN} N!} \right) + \frac{d}{2} N k_B \quad (101)$$

The internal energy is found by combining (31) and (32), so that,

$$U = F + TS = \frac{d}{2}Nk_B T \quad (102)$$

The pressure is given by,

$$P = - \left(\frac{\partial F}{\partial L^d} \right)_{T,N} = k_B T \frac{N}{L^d} = \frac{k_B N T}{L^d}, \quad (103)$$

which is the ideal gas law, while the chemical potential is,

$$\mu = \left(\frac{\partial F}{\partial N} \right)_{T,L^d} = k_B T \ln(\lambda^d N / L^d) \quad (104)$$

Assigned Problem 20b. Derive expressions for Ξ , PV , N/V and U for the Fermi gas in one and two dimensions. Write general expressions that are valid in any dimension. Find the leading order terms in the high temperature expansions for these quantities. Are the results what you expect? How do they compare with the result in three dimensions, and with the classical gas.

Solution The relations for the non-relativistic Fermi gas are,

$$\frac{P}{k_B T} = \frac{1}{\lambda^d} f_{d/2+1}(z); \quad \frac{N}{L^d} = \frac{1}{\lambda^d} f_{d/2}(z); \quad \frac{U}{L^d} = \frac{d k_B T}{2 \lambda^d} f_{d/2+1}(z) \quad (105)$$

These results are found from the integrals,

$$\ln(\Xi_F) = \frac{L}{2\pi\hbar} \int_0^\infty 2dp \ln(1 + ze^{-\beta p^2/2m}); \quad 1 - dimension \quad (106)$$

and

$$\ln(\Xi_F) = \left(\frac{L}{2\pi\hbar} \right)^2 \int_0^\infty 2\pi p dp \ln(1 + ze^{-\beta p^2/2m}) \quad 2 - dimension \quad (107)$$

These integrals and the analogous equations for N and V are expanded as in the three dimensional case. Following the procedure given in the solution to problem 7, we have,

$$\frac{PL}{k_B T} = \ln(\Xi_F) = \frac{L}{2\pi\hbar} \left(\frac{2m}{\beta} \right)^{1/2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l} \int_0^\infty 2dx e^{-x^2 l}; \quad 1 - dimension \quad (108)$$

and

$$\frac{PL^2}{k_B T} = \ln(\Xi_F) = \left(\frac{L}{2\pi\hbar} \right)^2 \left(\frac{2m}{\beta} \right) \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l} \int_0^\infty 2\pi x dx e^{-x^2 l}; \quad 2 - dimensions \quad (109)$$

which reduce to the expression for $P/k_B T$ given in Eq. (42). The leading order expansion of $f_{d/2}(z)$ at high temperature gives the chemical potential of the ideal classical gas in d dimensions so we recover the classical gas in d dimensions at sufficiently high temperatures.

Assigned Problem 21. Discuss the behavior of the Fermi gas at zero temperature in one and two dimensions. Is there different behavior as a function of dimension? Explain your reasoning.

Solution. We calculate the energy and degeneracy pressure to see if there is a dependence on dimension. We only carry out the ground state calculation. The Fermi wavevector in one, two and three dimensions is given by,

$$k_{F1} = \frac{\pi N}{L}; \quad k_{F2} = \left(\frac{4\pi N}{L^2} \right)^{1/2}; \quad k_{F3} = \left(\frac{6\pi^2 N}{V} \right)^{1/3} \quad (110)$$

so the Fermi energy is given by,

$$\epsilon_{F1} = \frac{\hbar^2}{2mL^2}(\pi N)^2; \quad \epsilon_{F2} = \frac{\hbar^2}{2mL^2}(4\pi N); \quad \epsilon_{F3} = \frac{\hbar^2}{2mL^2}(6\pi^2 N)^{2/3}; \quad (111)$$

The internal energy is given by,

$$U = \left(\frac{L}{2\pi}\right)^d \int d^d k \frac{\hbar^2 k^2}{2m} \quad (112)$$

In one two and three dimensions we find,

$$U_1 = \frac{1}{3}NE_{F1}; \quad U_2 = \frac{1}{2}NE_{F2}; \quad U_3 = \frac{3}{5}NE_{F3} \quad (113)$$

The degeneracy pressure is given by,

$$P = - \left(\frac{\partial U}{\partial L^d} \right)_{N,T} \quad (114)$$

Since E_F is proportional to $1/L^2$ doing the derivative with respect to L , L^2 and L^3 in one two and three dimensions, leads to the following expressions for the degeneracy pressure,

$$P_1 = \frac{2}{3L}NE_{F1}; \quad P_2 = \frac{1}{2L^2}NE_{F2}; \quad U_3 = \frac{2}{5L^3}NE_{F3}. \quad (115)$$

For fixed number of particles, the degeneracy pressure and internal energy are much higher for the one and two dimensional cases, first because E_F grows much more rapidly with N as the dimension is reduced and second because the prefactor grows more slowly with L as the dimension is reduced. This results are expected as particles are more confined in one and two dimensions, so the effect of Pauli exchange is stronger, so the total energy is expected to grow more rapidly in lower dimension and the degeneracy pressure should be higher.

Assigned Problem 22. Show that a d – dimensional Fermi gas with dispersion relation $\epsilon_p = cp^s$ obeys the relation,

$$P = \frac{s}{d} \frac{U}{L^d} \quad (116)$$

Solution. The equations for a Fermi gas with this dispersion relation in d dimensions is written as,

$$\frac{P}{k_B T} = \frac{c_d}{h^d} \int_0^\infty dp p^{d-1} \ln(1 + ze^{-\beta cp^s}); \quad \frac{U}{L^d} = \frac{c_d}{h^d} \int_0^\infty dp p^{d-1} (cp^s) \frac{ze^{-\beta cp^s}}{1 + ze^{-\beta cp^s}} \quad (117)$$

Integrating the pressure equation by parts gives,

$$I_1 = \int_0^\infty dp p^{d-1} \ln(1 + ze^{-\beta cp^s}) = \frac{1}{d} p^d \ln(1 + ze^{-\beta cp^s}) \Big|_0^\infty + \frac{s\beta}{d} \int_0^\infty dp p^{d-1} (cp^s) \frac{ze^{-\beta cp^s}}{1 + ze^{-\beta cp^s}} \quad (118)$$

The first term on the right hand side is zero and comparison of the remaining integral with the energy equation proves the relation.

Quiz Problem 23. Find the leading order term in the temperature dependence of the internal energy and specific heat of an three dimensional ultrarelativistic Fermi gas at low temperature.

Solution. The equations for U and N for the three-dimensional ultra-relativistic Fermi gas are,

$$N = 4\pi \left(\frac{L}{h}\right)^3 \int_0^\infty dp p^2 \frac{ze^{-\beta pc}}{1 + ze^{-\beta pc}} = \frac{4\pi}{(\beta c)^3} \left(\frac{L}{h}\right)^3 \int_0^\infty dx x^2 \frac{ze^{-x}}{1 + ze^{-x}} \quad (119)$$

and

$$U = 4\pi c \left(\frac{L}{h}\right)^3 \int_0^\infty dp p^3 \frac{ze^{-\beta pc}}{1 + ze^{-\beta pc}} = \frac{4\pi c}{(\beta c)^4} \left(\frac{L}{h}\right)^3 \int_0^\infty dx x^3 \frac{ze^{-x}}{1 + ze^{-x}} \quad (120)$$

We may expand the integral at small z , but this is not useful at low temperature. Instead we carry out the Sommerfeld expansion. Here we write it in more general form, generalizing Eq. (II.73) to,

$$I_s = \int_0^\infty dx x^{s-1} \frac{ze^{-x}}{1 + ze^{-x}} = \int_0^\infty dx x^{s-1} \frac{1}{e^{x-\nu} + 1} = \frac{1}{s} \int_0^\infty dx x^s \frac{e^{x-\nu}}{(e^{x-\nu} + 1)^2} \quad (121)$$

Expanding x^s about ν we have,

$$x^s = (\nu + (x - \nu))^s = f(0) + (x - \nu)f'(0) + \frac{1}{2}(x - \nu)^2 f''(0) + \dots \quad (122)$$

where $f(y) = (\nu + y)^s$, so that $f(0) = \nu^s$, $f'(0) = s\nu^{s-1}$, $f''(0) = s(s-1)\nu^{s-2}$. so that

$$x^s = (\nu + (x - \nu))^s = \nu^s + (x - \nu)s\nu^{s-1} + \frac{1}{2}(x - \nu)^2 s(s-1)\nu^{s-2} + \dots \quad (123)$$

Following the procedure of Eq. (II.77) and (II.78), we then have,

$$I_s = \frac{1}{s} [\nu^s I_0 + s\nu^{s-1} I_1 + \frac{1}{2}s(s-1)\nu^{s-2} I_2 + \dots]; \quad \text{where} \quad I_n = \int_{-\infty}^\infty dt \frac{t^n e^t}{(e^t + 1)^2} \quad (124)$$

$I_0 = 1$, while by symmetry I_n is zero for odd n . For even $n > 0$, I_n is related to the Reimann zeta function, through,

$$I_n = 2n(1 - 2^{1-n})(n-1)!\zeta(n), \quad \text{with} \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945} \quad (125)$$

Since the odd integral $I_0 = 1$, $I_1 = 0$, $I_2 = \pi^2/3$, we find,

$$I_s = \frac{1}{s} [\nu^s + \frac{\pi^2}{6}s(s-1)\nu^{s-2} + \dots] \quad (126)$$

Up to a prefactor that is defined differently here, the expansion above is consistent with Eqs. (II.78) and (II.88) as they must be. The expansions we need are then,

$$\frac{N}{V} = \frac{4\pi}{(h\beta c)^3} \frac{1}{3} [\nu^3 + \pi^2\nu + \dots] \quad (127)$$

and

$$\frac{U}{V} = \frac{4\pi c}{h^3(\beta c)^4} \frac{1}{4} [\nu^4 + 2\pi^2\nu^2 + \dots] \quad (128)$$

The leading order term in the expansion of the chemical potential is found using,

$$\frac{N}{V} = \frac{4\pi}{(h\beta c)^3} \frac{1}{3} (\beta\mu_0)^3 \quad \text{so} \quad \beta\mu_0 = h\beta c \left(\frac{3N}{4\pi V}\right)^{1/3} \quad (129)$$

The next correction is found using,

$$\nu_0^3 = (\nu_1)^3 + \pi^2\nu_0; \quad \text{so that} \quad \nu_1 = \nu_0 \left(1 - \frac{\pi^2}{3(\nu_0)^2}\right) = \beta\epsilon_F \left[1 - \frac{\pi^2}{3} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots\right]. \quad (130)$$

where $\epsilon_F = \mu_0 = \nu_0/\beta$. The internal energy expansion is,

$$\frac{U}{V} = \frac{\pi c}{h^3(\beta c)^4} \nu^4 \left[1 + \frac{2}{3}\pi^2\nu^{-2} + \dots\right] \approx \frac{\pi c}{h^3(\beta c)^4} \beta\epsilon_F^4 \left[1 - \frac{4\pi^2}{3} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots\right] \left[1 + 2\pi^2\beta\epsilon_F^{-2} + \dots\right] \quad (131)$$