PHYS851 Quantum Mechanics I, Fall 2009

HOMEWORK ASSIGNMENT 3: Solutions
Fundamentals of Quantum Mechanics

1. [10pts] The trace of an operator is defined as $Tr\{A\} = \sum_m \langle m|A|m\rangle$, where $\{|m\rangle\}$ is a suitable basis set.

   (a) Prove that the trace is independent of the choice of basis.

   Answer:
   Let $\{|m\rangle\}$ and $\{|e_m\rangle\}$ be two independent basis sets for our Hilbert space.
   We must show that $\sum_m \langle e_m|A|e_m\rangle = \sum_m \langle m|A|m\rangle$.
   Proof:

$$\begin{aligned}
\sum_m \langle e_m|A|e_m\rangle &= \sum_{mm'm''} \langle e_m|m'\rangle\langle m'|A|m''\rangle\langle m''|e_m\rangle && (1)\\
&= \sum_{mm'm''} \langle m''|e_m\rangle\langle e_m|m'\rangle\langle m'|A|m''\rangle && (2)\\
&= \sum_{m'm''} \langle m''|m'\rangle\langle m'|A|m''\rangle && (3)\\
&= \sum_{m'm''} \delta_{m'm''}\langle m'|A|m''\rangle && (4)\\
&= \sum_{m'} \langle m'|A|m'\rangle && (5)\\
&= \sum_{m} \langle m|A|m\rangle && (6)
\end{aligned}$$

   (b) Prove the linearity of the trace operation by proving $Tr\{aA + bB\} = aTr\{A\} + bTr\{B\}$.

   Answer:

$$\begin{aligned}
Tr\{aA + bB\} &= \sum_m \langle m|aA + bB|m\rangle && (7)\\
&= \sum_m \left(a\langle m|A|m\rangle + b\langle m|B|m\rangle\right) && (8)\\
&= a\sum_m \langle m|A|m\rangle + b\sum_m \langle m|B|m\rangle && (9)\\
&= a\,Tr\{A\} + b\,Tr\{B\} && (10)
\end{aligned}$$

(c) Prove the cyclic property of the trace by proving $Tr\{ABC\} = Tr\{BCA\} = Tr\{CAB\}$.

Answer:
First, if $Tr\{ABC\} = Tr\{BCA\}$ then it follows that $Tr\{BCA\} = Tr\{CAB\}$, so we need only prove the first identity.

$$\begin{aligned} Tr\{ABC\} &= \sum_m \langle m|ABC|m\rangle && (11) \\ &= \sum_{mm'm''} \langle m|A|m'\rangle\langle m'|B|m''\rangle\langle m''|C|m\rangle && (12) \\ &= \sum_{mm'm''} \langle m''|C|m\rangle\langle m|A|m'\rangle\langle m'|B|m''\rangle && (13) \\ &= Tr\{CAB\} && (14) \end{aligned}$$

2. Consider the system with three physical states $\{|1\rangle, |2\rangle, |3\rangle\}$. In this basis, the Hamiltonian matrix is:

$$H = \begin{pmatrix} 1 & 2i & 1 \\ -2i & 2 & -2i \\ 1 & 2i & 1 \end{pmatrix} \tag{15}$$

Find the eigenvalues $\{\omega_1, \omega_2, \omega_3\}$ and eigenvectors $\{|\omega_1\rangle, |\omega_2\rangle, |\omega_3\rangle\}$ of H. Assume that the initial state of the system is $|\psi(0)\rangle = |1\rangle$. Find the three components $\langle 1|\psi(t)\rangle$, $\langle 2|\psi(t)\rangle$, and $\langle 3|\psi(t)\rangle$. Give all of your answers in proper Dirac notation.

Answer:
The eigenvalues are solutions to

$$\det |H - \hbar\omega I| = 0 \tag{16}$$

Taking the determinate in Mathematica gives

$$4\omega + 4\omega^2 - \omega^3 = 0 \tag{17}$$

which factorizes as

$$\omega(\omega^2 - 4\omega - 4) = 0 \tag{18}$$

which has as its solutions

$$\omega_1 = 2(1 - \sqrt{2}) \tag{19}$$

$$\omega_2 = 0 \tag{20}$$

$$\omega_3 = 2(1 + \sqrt{2}) \tag{21}$$

the corresponding eigenvectors are

$$|\omega_1\rangle = \frac{1}{2}(|1\rangle + \sqrt{2}i|2\rangle + |3\rangle) \tag{22}$$

$$|\omega_2\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |3\rangle) \tag{23}$$

$$|\omega_3\rangle = \frac{1}{2}(|1\rangle - \sqrt{2}i|2\rangle + |3\rangle) \tag{24}$$

The components of $|\psi(t)\rangle$ are found via $|\psi(t)\rangle = e^{-iHt}|\psi(0)|\rangle$, giving

$$\langle 1|\psi(t)\rangle = \frac{1}{4}\left(2 + e^{-i2(1-\sqrt{2})t} + e^{-i2(1+\sqrt{2})t}\right) \tag{25}$$

$$\langle 2|\psi(t)\rangle = \frac{i}{2\sqrt{2}}\left(e^{-i2(1-\sqrt{2})t} - e^{-i2(1+\sqrt{2})t}\right) \tag{26}$$

$$\langle 3|\psi(t)\rangle = \frac{1}{4}\left(-2 + e^{-i2(1-\sqrt{2})t} + e^{-i2(1+\sqrt{2})t}\right) \tag{27}$$

The mathematic script I used to work this problem is on the following page:

```
In[6]:= H = {{1, 2 I, 1}, {-2 I, 2, -2 I}, {1, 2 I, 1}};

In[7]:= MatrixForm[H]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 1 & 2i & 1 \\ -2i & 2 & -2i \\ 1 & 2i & 1 \end{pmatrix}$$

```
In[9]:= Det[H - ω IdentityMatrix[3]]
```

Out[9]= $4\omega + 4\omega^2 - \omega^3$

```
In[12]:= FullSimplify[Eigensystem[H]]
```

Out[12]= $\left\{\left\{2\left(1+\sqrt{2}\right),\ 2-2\sqrt{2},\ 0\right\},\ \left\{\left\{1,\ -i\sqrt{2},\ 1\right\},\ \left\{1,\ i\sqrt{2},\ 1\right\},\ \{-1,\ 0,\ 1\}\right\}\right\}$

```
In[13]:= ψ0 = {1, 0, 0};

In[14]:= MatrixForm[ψ0]
```

Out[14]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

```
In[17]:= ψt = FullSimplify[MatrixExp[-i H t].ψ0];

In[18]:= MatrixForm[ψt]
```

Out[18]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}\left(2 + e^{2\left(-1+\sqrt{2}\right)it} + e^{-2\left(1+\sqrt{2}\right)it}\right) \\ \frac{i\, e^{-2\left(1+\sqrt{2}\right)it}\left(-1+e^{4\sqrt{2}\,it}\right)}{2\sqrt{2}} \\ \frac{1}{4}\left(-2 + e^{2\left(-1+\sqrt{2}\right)it} + e^{-2\left(1+\sqrt{2}\right)it}\right) \end{pmatrix}$$

3. Cohen-Tannoudji: pp 203-206: problems 2.2, 2.6, 2.7

2.2 (a) The operator σ_y is hermitian:

$$\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \tag{28}$$

We find the eigenvalues via $\det|\sigma_y - \omega I| = 0$:

$$\det \begin{vmatrix} -\omega & -i \\ i & -\omega \end{vmatrix} = \omega^2 - 1 = 0 \tag{29}$$

The solutions are $\omega = 1$ and $\omega = -1$.
Let the corresponding eigenvectors be $|+\rangle$ and $|-\rangle$, so that

$$\sigma_y|\pm\rangle = \pm|\pm\rangle. \tag{30}$$

Hit this equation with the bra $\langle 1|$ and insert the projector onto the $\{|1\rangle, |2\rangle\}$ basis:

$$(-\langle 1|\sigma_y|1\rangle \pm 1)\langle 1|\pm\rangle - \langle 1|\sigma_y|2\rangle\langle 2|\pm\rangle = 0 \tag{31}$$

inserting the values of the matrix elements of σ_y then gives:

$$\pm\langle 1|\pm\rangle + i\langle 2|\pm\rangle = 0 \tag{32}$$

a non-normalized solution is then

$$\langle 1|\pm\rangle = i \tag{33}$$

$$\langle 2|\pm\rangle = \mp 1 \tag{34}$$

the normalized eigenvectors are then given, up to arbitrary overall phase-factors, by

$$|+\rangle = \frac{1}{\sqrt{2}}(i|1\rangle - |2\rangle) \tag{35}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle) \tag{36}$$

(b) The projectors are given by $I_\pm = |\pm\rangle\langle\pm|$. In matrix form, in the $\{|1\rangle, |2\rangle\}$ basis, these are

$$I_\pm = \begin{pmatrix} \langle 1|\pm\rangle\langle\pm|1\rangle & \langle 1|\pm\rangle\langle\pm|2\rangle \\ \langle 2|\pm\rangle\langle\pm|1\rangle & \langle 2|\pm\rangle\langle\pm|2\rangle \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} \frac{i}{\sqrt{2}}\frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}}\frac{\mp 1}{\sqrt{2}} \\ \frac{\mp 1}{\sqrt{2}}\frac{-i}{\sqrt{2}} & \frac{\mp 1}{\sqrt{2}}\frac{\mp 1}{\sqrt{2}} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} \frac{1}{2} & \mp\frac{i}{2} \\ \pm\frac{i}{2} & \frac{1}{2} \end{pmatrix} \tag{39}$$

we need to show that $I_\pm^2 = I_\pm$ and $I_+ + I_- = I$:

$$I_\pm^2 = \begin{pmatrix} \frac{1}{2} & \frac{\mp i}{2} \\ \frac{\pm i}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\mp i}{2} \\ \frac{\pm i}{2} & \frac{1}{2} \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} \frac{1}{4} + \frac{1}{4} & \mp\frac{i}{4} \mp \frac{i}{4} \\ \pm\frac{i}{4} \pm \frac{i}{4} & \frac{1}{4} + \frac{1}{4} \end{pmatrix} \tag{41}$$

$$= \begin{pmatrix} \frac{1}{2} & \mp\frac{i}{2} \\ \pm\frac{i}{2} & \frac{1}{2} \end{pmatrix} \tag{42}$$

$$= I_\pm \tag{43}$$

$$
\begin{aligned}
I_+ + I_- &= \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ +\frac{i}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & +\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} & (44) \\
&= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & -\frac{i}{2} + \frac{i}{2} \\ +\frac{i}{2} - \frac{i}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} & (45) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (46) \\
&= I & (47)
\end{aligned}
$$

(c) The results for M and L_y are attached.

```
(* Enter the matrix M *)
```

In[190]:=

```
M = {{2, I Sqrt[2]}, {-I Sqrt[2], 3}};
MatrixForm[M]
```

Out[191]//MatrixForm=

$$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}$$

```
(* now compute the Hermitian conjugate and make sure it is self-adjoint *)
```

In[192]:=

```
MatrixForm[Conjugate[Transpose[M]]]
```

Out[192]//MatrixForm=

$$\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}$$

```
(* Now find the eigenvalues *)
```

In[198]:=

```
y = Solve[Det[M - ω IdentityMatrix[2]] == 0];
```

```
(* extract the eigenvalues from y *)
```

In[213]:=

```
ω1 = ω /. y[[1, 1]]
```

Out[213]=

1

In[214]:=

```
ω2 = ω /. y[[2, 1]]
```

Out[214]=

4

```
(* now we find the eigenvectors the old-fashioned way*)
(* e1 and e2 will be the normalized eigenvectors *)
```

In[227]:=

```
y = Solve[((M - ω1 IdentityMatrix[2]).{1, c})[[1]] == 0, c]
```

Out[227]=

$$\left\{\left\{c \to \frac{i}{\sqrt{2}}\right\}\right\}$$

In[229]:=

$$\text{e1} = \frac{\{1,\ c\ /.\ y[[1]]\}}{\text{Sqrt}[\text{Conjugate}[\{1,\ c\ /.\ y[[1]]\}].\{1,\ c\ /.\ y[[1]]\}]}$$

Out[229]=

$$\left\{\sqrt{\frac{2}{3}},\ \frac{i}{\sqrt{3}}\right\}$$

In[230]:=

```
y = Solve[((M - ω2 IdentityMatrix[2]).{1, c})[[1]] == 0, c]
```

Out[230]=

$\{\{c \to -i\sqrt{2}\}\}$

In[231]:=

```
e2 = {1, c /. y[[1]]} / Sqrt[Conjugate[{1, c /. y[[1]]}].{1, c /. y[[1]]}]
```

Out[231]=

$\left\{\frac{1}{\sqrt{3}}, -i\sqrt{\frac{2}{3}}\right\}$

```
(* we can form the projectors via the outer-product function *)
```

In[243]:=

```
I1 = Outer[Times, Conjugate[e1], e1];
MatrixForm[I1]
```

Out[244]//MatrixForm=

$$\begin{pmatrix} \frac{2}{3} & \frac{i\sqrt{2}}{3} \\ -\frac{i\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

In[245]:=

```
I2 = Outer[Times, Conjugate[e2], e2];
MatrixForm[I2]
```

Out[246]//MatrixForm=

$$\begin{pmatrix} \frac{1}{3} & -\frac{i\sqrt{2}}{3} \\ \frac{i\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}$$

```
(* To verify the othogonality relation, we compute Conjugate[e1].e2  *)
```

In[251]:=

```
Conjugate[e1].e2
```

Out[251]=

0

```
(* Lastly, we verify the closure relation *)
```

In[252]:=

```
MatrixForm[I1 + I2]
```

Out[252]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
(* First we enter the matrix L_y *)
```

$$Ly = \frac{\hbar}{2\,I}\{\{0, Sqrt[2], 0\}, \{-Sqrt[2], 0, Sqrt[2]\}, \{0, -Sqrt[2], 0\}\};$$

```
MatrixForm[Ly]
```

$$\begin{pmatrix} 0 & -\frac{i\,\hbar}{\sqrt{2}} & 0 \\ \frac{i\,\hbar}{\sqrt{2}} & 0 & -\frac{i\,\hbar}{\sqrt{2}} \\ 0 & \frac{i\,\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
(* Now we take the transpose and complex conjugate *)
```

$$MatrixForm[Conjugate[Transpose[Ly]]] \;/.\; Conjugate[\hbar] \to \hbar$$

$$\begin{pmatrix} 0 & -\frac{i\,\hbar}{\sqrt{2}} & 0 \\ \frac{i\,\hbar}{\sqrt{2}} & 0 & -\frac{i\,\hbar}{\sqrt{2}} \\ 0 & \frac{i\,\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
(* By comparison, we see it is Hermitian *)
```

```
(* This the the matrix whose determinant gives the spectrum of L_y *)
```

$$MatrixForm[Ly - \omega\, IdentityMatrix[3]]$$

$$\begin{pmatrix} -\omega & -\frac{i\,\hbar}{\sqrt{2}} & 0 \\ \frac{i\,\hbar}{\sqrt{2}} & -\omega & -\frac{i\,\hbar}{\sqrt{2}} \\ 0 & \frac{i\,\hbar}{\sqrt{2}} & -\omega \end{pmatrix}$$

```
(* Here we let mathematica solve the characteristic polynomial *),
```

$$y = Solve[Det[Ly - \omega\, IdentityMatrix[3]] == 0];$$

```
Solve::svars : Equations may not give solutions for all "solve" variables. More…
```

```
(* so the eigenvalues are: *)
```

```
ω1 = ω /. y[[1, 1]]
ω2 = ω /. y[[2, 1]]
ω3 = ω /. y[[3, 1]]
```

0

$-\hbar$

$\hbar$

```
(* "Eigensystem" will give the eigenvalues and (un-normalized eigenvectors *)

y = Eigensystem[Ly];

e1p = y[[2, 1]];
e2p = y[[2, 2]];
e3p = y[[2, 3]];

(* To normalize them, we compute the normalization constants *)

n1 = Sqrt[Conjugate[e1p].e1p];
n2 = Sqrt[Conjugate[e2p].e2p];
n3 = Sqrt[Conjugate[e3p].e3p];

(* The normalized eigenvectors are then: *)

e1 = e1p / n1
e2 = e2p / n2
e3 = e3p / n3
```

$\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

$\left\{-\frac{1}{2}, \frac{i}{\sqrt{2}}, \frac{1}{2}\right\}$

$\left\{-\frac{1}{2}, -\frac{i}{\sqrt{2}}, \frac{1}{2}\right\}$

```
(* Here we form the projectors by using Mathematicas outer-product function *)

I1 = Outer[Times, Conjugate[e1], e1];
I2 = Outer[Times, Conjugate[e2], e2];
I3 = Outer[Times, Conjugate[e3], e3];
```

```
MatrixForm[I1]
MatrixForm[I2]
MatrixForm[I3]
```

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & -\frac{i}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{i}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & \frac{i}{2\sqrt{2}} & -\frac{1}{4} \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2} & \frac{i}{2\sqrt{2}} \\ -\frac{1}{4} & -\frac{i}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

(* Here we square the matrices. By comparison we see that $I_j{}^2 = I_j$ *)

```
MatrixForm[MatrixPower[I1, 2]]
MatrixForm[MatrixPower[I2, 2]]
MatrixForm[MatrixPower[I3, 2]]
```

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & -\frac{i}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{i}{2\sqrt{2}} & \frac{1}{2} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{i}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & \frac{i}{2\sqrt{2}} & -\frac{1}{4} \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2} & \frac{i}{2\sqrt{2}} \\ -\frac{1}{4} & -\frac{i}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

```
(* Lastly, we sum the projectors to verify the closure relation *)

MatrixForm[I1 + I2 + I3]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.6 Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{48}$$

By definition we have

$$e^{i\alpha\sigma_x} = \sum_{m=0}^{\infty} \frac{(i\alpha)^m}{m!}\sigma_x^m \tag{49}$$

For the $m = 0$ term, we have $\sigma_x^0 = I$, and for the $m = 1$ term, $\sigma_x^1 = \sigma_x$. For the $m = 2$ term, we find

$$\begin{aligned} \sigma_x^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && (50) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (51) \\ &= I && (52) \end{aligned}$$

From this it follows that $\sigma_x^3 = \sigma_x$ and $\sigma_x^4 = I$, and so on. So we see that all odd powers give σ_x and all even powers give I. Thus we can write

$$\begin{aligned} e^{i\alpha\sigma_x} &= I\sum_{m=0}^{\infty} \frac{(i\alpha)^{2m}}{(2m)!} + \sigma_x \sum_{m=0}^{\infty} \frac{(i\alpha)^{2m+1}}{(2m+1)!} && (53) \\ &= I\sum_{m=0}^{\infty} \frac{(-1)^m(\alpha)^{2m}}{(2m)!} + i\sigma_x \sum_{m=0}^{\infty} \frac{(-1)^m(\alpha)^{2m+1}}{(2m+1)!} && (54) \\ &= I\cos(\alpha) + i\sigma_x \sin(\alpha) && (55) \end{aligned}$$

where the last step is possible because we recognize the series expansions for sin and cos.

2.7 We can start by computing σ_y^2 and seeing what happens

$$\begin{aligned}
\sigma_y^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && (56)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (57)\\
&= I && (58)
\end{aligned}$$

Since the previous derivation followed strictly from $\sigma_x^2 = I$, then the same result must be valid for σ_y.

$$e^{i\alpha\sigma_y} = I\cos(\alpha) + i\sigma_y \sin(\alpha) \tag{59}$$

For $\sigma_u = \lambda\sigma_x + \mu\sigma_y$, here we start by computing σ_u:

$$\sigma_u = \begin{pmatrix} 0 & \lambda - i\mu \\ \lambda + i\mu & 0 \end{pmatrix} \tag{60}$$

Computing the square gives

$$\begin{aligned}
\sigma_u^2 &= \begin{pmatrix} 0 & \lambda - i\mu \\ \lambda + i\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda - i\mu \\ \lambda + i\mu & 0 \end{pmatrix} && (61)\\
&= \begin{pmatrix} \lambda^2 + \mu^2 & 0 \\ 0 & \lambda^2 + \mu^2 \end{pmatrix} && (62)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (63)\\
&= I && (64)
\end{aligned}$$

Thus it follows immediately that

$$e^{i\alpha\sigma_u} = I\cos(\alpha) + i\sigma_u \sin(\alpha) \tag{65}$$

$$e^{2i\sigma_x} = I\cos(2) + i\sigma_x \sin(2) = \begin{pmatrix} -0.42 & 0.91i \\ 0.91i & -0.42 \end{pmatrix} \tag{66}$$

$$e^{i\sigma_x} = I\cos(1) + i\sigma_x \sin(1) = \begin{pmatrix} 0.54 & 0.84i \\ 0.84i & 0.54 \end{pmatrix} \tag{67}$$

$$\begin{aligned}
\left(e^{i\sigma_x}\right)^2 &= = \begin{pmatrix} 0.54 & 0.84i \\ 0.84i & 0.54 \end{pmatrix} \begin{pmatrix} 0.54 & 0.84i \\ 0.84i & 0.54 \end{pmatrix} && (68)\\
&= \begin{pmatrix} -0.42 & 0.91i \\ 0.91i & -0.42 \end{pmatrix} && (69)\\
&= e^{2i\sigma_x} && (70)
\end{aligned}$$

$$\begin{aligned}
e^{i(\sigma_x+\sigma_y)} &= \exp\left[\begin{pmatrix} 0 & i+1 \\ i-1 & 0 \end{pmatrix}\right] && (71)\\
&= \begin{pmatrix} 0.16 & 0.70 + 0.70i \\ -0.70 + 0.70i & 0.16 \end{pmatrix} && (72)
\end{aligned}$$

$$e^{i\sigma_y} = \begin{pmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{pmatrix} \tag{73}$$

$$\begin{aligned} e^{i\sigma_x} e^{i\sigma_y} &= \begin{pmatrix} 0.54 & 0.84i \\ 0.84i & 0.54 \end{pmatrix} \begin{pmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{pmatrix} && (74) \\ &= \begin{pmatrix} 0.29 - 0.71i & 0.45 + 0.45i \\ -0.45 + 0.45i & 0.29 + 0.71i \end{pmatrix} && (75) \\ &\neq e^{i(\sigma_x + \sigma_y)} && (76) \end{aligned}$$

4. Cohen-Tannoudji ;pp341-350: problem 3.14

   a. From inspection, the eigenvalues of H are $\hbar\omega_0$ and $2\hbar\omega_0$, with the latter being doubly degenerate. The eigenstates are $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$, with the first being the non-degenerate state. Thus a measurement of the energy would yield $\hbar\omega_0$ with probability 1/2 and $2\hbar\omega_0$ with probability 1/2.

$$\langle H\rangle = \hbar\omega_0\left(1\frac{1}{2}+2\frac{1}{2}\right) = \frac{3}{2}\hbar\omega_0$$

$$\langle H^2\rangle = \hbar^2\omega_0^2\left(1\frac{1}{2}+4\frac{1}{2}\right) = \frac{5}{2}\hbar^2\omega^2$$

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \hbar\omega_0\sqrt{\frac{5}{2}-\frac{9}{4}} = \frac{1}{2}\hbar\omega_0$$

   b. The eigenvalues of A are a and $-a$, with eigenstates $|a,1\rangle = |u_1\rangle$, $|a,2\rangle = \frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle)$, and $|-a\rangle = \frac{1}{\sqrt{2}}(||u_2\rangle - |u_3\rangle)$

      Clearly, the initial state is a superposition of $|a,1\rangle$ and $|a,2\rangle$, so the probability to obtain a is 1 and to obtain $-a$ is 0. Thus after the measurement, the state will remain unchanged.

   c.
$$|\psi(t)\rangle = \frac{e^{-i\omega_0 t}}{\sqrt{2}}|u_1\rangle + \frac{e^{-i2\omega_0 t}}{2}|u_2\rangle + \frac{e^{-i2\omega_0 t}}{2}|u_3\rangle$$

   d. Because the initial state is a superposition of degenerate eigenstates of A, we know that

$$\langle A\rangle(t) = a$$

$$\begin{aligned} B|\psi(t)\rangle &= b\left(|u_1\rangle\langle u_2| + |u_2\rangle\langle u_1| + |u_3\rangle\langle u_3|\right)\left(\frac{e^{-i\omega_0 t}}{\sqrt{2}}|u_1\rangle + \frac{e^{-i2\omega_0 t}}{2}|u_2\rangle + \frac{e^{-i2\omega_0 t}}{2}|u_3\rangle\right) \\ &= b\left(\frac{e^{-i2\omega_0 t}}{2}|u_1\rangle + \frac{e^{-i\omega_0 t}}{\sqrt{2}}|u_2\rangle + \frac{e^{-i2\omega_0 t}}{2}|u_3\rangle\right) \end{aligned}$$

      so that
$$\langle\psi(t)|B|\psi(t)\rangle = b\left(\frac{e^{-i\omega_0 t}}{2\sqrt{2}} + \frac{e^{i\omega_0 t}}{2\sqrt{2}} + \frac{1}{4}\right) = b\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\cos(\omega_0 t)\right)$$

   e. From the previous problem, we can see that a measurement of A at time t will yield a. The eigenvalues of B are b and $-b$, with eigenstates $|b,1\rangle = |u_3\rangle$, $|b,2\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, and $|-b\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle)$.
      Projecting $|\psi(t)\rangle$ onto $|-b\rangle$ gives $\langle -b|\psi(t)\rangle = \frac{e^{-i\omega_0 t}}{2} + \frac{e^{-i2\omega_0 t}}{2\sqrt{2}}$
      Taking the square modulus gives $p(-b) = \frac{3}{8} + \frac{\sqrt{2}}{4}\cos(\omega_0 t)$
      By probability conservation, we have then $p(b) = 1 - p(-b) = \frac{5}{8} - \frac{\sqrt{2}}{4}\cos(\omega_0 t)$