B. The Binomial Theorem

A general expression that we often encounter in algebra and calculus is $(A + B)^p$. A and B denote real numbers; the exponent p might be an integer, although not necessarily. The binomial theorem tells how to expand this expression in powers of A and B.

The simplest example is $p = 2$, which is familiar from school,

$$(A + B)^2 = A^2 + 2AB + B^2. \tag{B-1}$$ $\boxed{\texttt{eq:AB2}}$

For example, what is the square of $5 + 7$? We could first add, $5 + 7 = 12$, and then square, $12^2 = 144$. Or, we could use (B-1), $25 + 70 + 49 = 144$. For a case where the values of A and B are known, there is no particular advantage in the expansion. But if A or B (or both) are symbolic variables, expanding in powers in powers may lead to simplification.

Example 2. Simplify $(A + B)^2 - (A - B)^2$.

Solution. Using (B-1), the quantity is

$$\left(A^2 + 2AB + B^2\right) - \left(A^2 - 2AB + B^2\right) = 4AB. \tag{B-2}$$

We will also need higher powers, such as $(A + B)^3$ or $(A + B)^4$. The proof of (B-1) and its genrealizations is based on the associative property of multiplication,

$$(A + B)C = AC + BC. \tag{B-3}$$ $\boxed{\texttt{eq:assoc}}$

For example, $(3 + 4) \times 5 = 35$ is equal to $15 + 20$. Letting $C = (A + B)$ in (B-3) leads to (B-1):

$$\begin{aligned}
(A + B)(A + B) &= A(A + B) + B(A + B) \\
&= A^2 + AB + BA + B^2 \\
&= A^2 + 2AB + B^2.
\end{aligned} \tag{B-4}$$

Then letting $C = A^2 + 2AB + B^2$ leads to the equation for $(A + B)^3$,

$$\begin{aligned}
(A + B)^3 &= (A + B)(A^2 + 2AB + B^2) \\
&= A(A^2 + 2AB + B^2) + B(A^2 + 2AB + B^2) \\
&= A^3 + 3A^2B + 3AB^2 + B^3.
\end{aligned} \tag{B-5}$$ $\boxed{\texttt{eq:AB3}}$

The expansion for $(A + B)^4$ is derived similarly,

$$(A + B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4. \tag{B-6}$$ $\boxed{\texttt{eq:AB4}}$

a	b
c	d

Table B-1: Properties of factorials tbl1

A general theorem for $(A+B)^n$, with n an integer, is given in the next section. The result is extended to $(A+B)^p$ for noninteger p in the final section.

B.1 INTEGER POWERS

Theorem B-1. The expansion of $(A+B)^n$ in powers of A and B, where n is a positive integer, is

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^{n-k} B^k. \qquad \text{(B-7)}$$ eq:BT

The sum in (B-7) has $n+1$ terms. Each term is the product of a numerical constant, a power of A, and a power of B. The exponents of A and B add to n. The A and B powers are

$$A^n, \quad A^{n-1}B, \quad A^{n-2}B^2, \quad \ldots, \quad AB^{n-1}, \quad B^n,$$

i.e., all combinations such that the sum of exponents is n. For $n=2$ these are the three terms in (B-1); for $n=3$ these are the four terms in (B-5), and so on.

The coefficient of $A^{n-k}B^k$ is called the *binomial coefficient*, denoted by $\binom{n}{k}$, and defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \qquad \text{(B-8)}$$ eq:BC

Here $n!$ (read as "n factorial") is, for $n \geq 1$, the product of all the integers from 1 to n. Table B-1 lists some properties of factorials. The value of 0! is defined to be 1. Also, note the recursion relation $(n+1)! = (n+1)n!$.

The highest power of A in (B-7) is A^n (the term with $k=0$); the coefficient is

$$\binom{n}{0} = \frac{n!}{0!n!} = 1.$$

The highest power of B is B^n, which also has coefficient 1. Note how these results agree with (B-1), (B-5) and (B-6).

Proof of Theorem B-1. Equation (B-7) is proven by induction. It is obviously true for $n = 1$,

$$(A+B)^1 = \frac{1!}{0!1!}A^1B^0 + \frac{1!}{1!0!}A^0B^1 = A + B. \tag{B-9}$$

Now assume that it is true for n, and consider the next power, $n+1$:

$$\begin{aligned}(A+B)^{n+1} &= (A+B)(A+B)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(A^{n-k+1}B^k + A^{n-k}B^{k+1}\right). \end{aligned} \tag{B-10}$$

eq:ind1

The exponents sum to $n+1$ in both terms in the sum. Now rearrange the terms in the sum to the form in (B-7),

$$\sum_{\ell=0}^{n+1} C_\ell A^{n+1-\ell}B^\ell. \tag{B-11}$$

We use here a different summation variable ℓ so that it will not be confused with k in (B-10). The coefficient of the term $A^{n+1-\ell}B^\ell$ is

$$C_\ell = \binom{n}{\ell} + \binom{n}{\ell-1}; \tag{B-12}$$

eq:Cell

the two terms come from the two terms in (B-10). (The first term is the coefficient of $A^{n-k+1}B^k$ with $k = \ell$; the second term is the coefficient of $A^{n-k}B^{k+1}$ with $k = \ell - 1$.) C_ℓ can be simplified using properties of the factorial,

$$\begin{aligned} C_\ell &= \frac{n!}{\ell!(n-\ell)!} + \frac{n!}{(\ell-1)!(n-\ell+1)!} \\ &= \frac{n!}{(\ell-1)!(n-\ell)!}\left[\frac{1}{\ell} + \frac{1}{n-\ell+1}\right] \\ &= \frac{n!}{(\ell-1)!(n-\ell)!}\,\frac{n+1}{\ell(n+1-\ell)} \\ &= \frac{(n+1)!}{\ell!(n+1-\ell)!} = \binom{n+1}{\ell}; \end{aligned} \tag{B-13}$$

eq:proof

i.e., C_ℓ is the binomial coefficient for $n+1$ factors. Hence (B-7) is true for $(A+B)^{n+1}$. By induction, the theorem is proven.

n	binomial coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
	etc.

Table B-2: Pascal's triangle tbl2

Pascal's triangle

The coefficients of $A^{n-k}B^k$ in the expansion (B-7) may be arranged in Pascal's triangle, shown in Table B-2. For example, the numbers in the rows with $n = 2$, 3, and 4, agree with the coefficients in Eqs. (B-1), (B-5), and (B-6). In Pascal's triangle, each number (for $n > 0$) is the sum of the two adjacent numbers in the line above. In terms of binomial coefficients, this constuction is

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

which is just the identity in the proof of Theorem B-1 [see (B-12) and (B-13)].

B.2 THE BINOMIAL EXPANSION FOR NONINTEGER POWERS

Theorem B-1 is an exact and finite equation for any A and B and integer n. There is a related expression if n is not an integer, discovered by Isaac Newton.

Let p be a real number, positive or negative. Then consider $(A+B)^p \equiv N$. The binomial expansion, generalized to noninteger p, is

$$\begin{aligned}(A+B)^p &= A^p + \frac{p}{1!}A^{p-1}B + \frac{p(p-1)}{2!}A^{p-2}B^2 \\ &+ \frac{p(p-1)(p-2)}{3!}A^{p-3}B^3 + \cdots + C(p,k)A^{p-k}B^k + \cdots,\end{aligned} \tag{B-14}$$

eq:BE

the general coefficient (for $k > 0$) is

$$C(p,k) = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}. \tag{B-15}$$

In general the number of terms that must be summed in (B-14) is infinite, i.e., the expansion is an infinite series,

$$(A+B)^p = \sum_{k=0}^{\infty} C(p,k) A^{p-k} B^k. \tag{B-16}$$

eq:BE2

If $p = n$, an integer, then the coefficient of the term proportional to $A^{n-k}B^k$ is

$$C(n,k) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}, \tag{B-17}$$

eq:casepn

just the binomial coefficient for power n. In this case the number of terms in the expansion is finite, and equal to $n+1$. The coefficient $C(n,k)$ is 0 if $k > n$, because one of the factors in the numerator of (B-17) is 0. For example, $C(n,n+1) = 0$ because the final factor is $n-(n+1)+1 = 0$. Thus the binomial expansion (B-16) reduces to Theorem B-1 if p is an integer.

If p is not an integer then (B-16) is an infinite series. In order for the series to be convergent, A should be the larger (in magnitude) of A and B. (Otherwise, reverse the roles of A and B in the right-hand side of (B-16).) Then the kth terms is proportional to

$$A^{p-k}B^k = A^p \left(\frac{B}{A}\right)^k$$

where $|B/A|$ is less than 1. As k increases, the factor $(B/A)^k$ gets smaller and smaller (in magnitude) so that the sum can converge to a finite value as more and more terms are added.[1]

An interesting special case is $A = 1$ and $B = x$. Then the binomial expansion becomes

$$\begin{aligned}(1+x)^p &= 1 + \frac{p}{1!}x + \frac{p(p-1)}{2!}x^2 \\ &\quad + \cdots + \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}x^k + \cdots \ . \end{aligned} \tag{B-18}$$

This series is the Taylor series (Chapter 7) of the function $(1+x)^p$.

Example 4. Estimate $\sqrt{5}$ from the binomial expansion.

Solution. Write $\sqrt{5} = (4+1)^{1/2}$, and apply (B-16) with $A = 4$, $B = 1$ and $p = 1/2$; that is,

$$\sqrt{5} \quad = \quad A^{1/2} \sum_{k=0}^{\infty} C\left(\frac{1}{2}, k\right) \left(\frac{B}{A}\right)^k$$

[1] Convergence of infinite series is discussed in Appendix C.

$$
\begin{aligned}
&= 2\left[1+\frac{1}{1!}\frac{1}{2}\left(\frac{1}{4}\right)-\frac{1}{2!}\frac{1}{4}\left(\frac{1}{4}\right)^2+\frac{1}{3!}\frac{3}{8}\left(\frac{1}{4}\right)^3+\cdots\right] \\
&\approx 2.236. \qquad \text{(B-19)}
\end{aligned}
$$

The first four terms in the series give an approximation to $\sqrt{5}$ that is accurate to 3 decimal places.

Example 6. Calculate $\sqrt[3]{3}$ accurate to 3 decimal places.

Solution. Write $\sqrt[3]{3} = (8-5)^{1/3} = 2(1-5/8)^{1/3}$. The coefficient $C(1/3,k)$ may be calculated from the recursion relation

$$C(1/3,k+1) = \frac{1/3-k}{k+1}C(1/3,k).$$

The first few coefficients are

$$1,\quad \frac{1}{3},\quad -\frac{1}{9},\quad \frac{5}{81},\quad \ldots\ .$$

The table shows how the expansion (B-14) converges as terms are added one by one. To achieve an accuracy of 3 decimal places, 12 terms in the sum are necessary, which gives $\sqrt[3]{3} = 1.442$.

n	sum of n terms
1	2
2	1.583
3	1.497
4	1.466
5	1.454
6	1.448
7	1.445
8	1.444
9	1.443
10	1.443
12	1.442

Example 8. Write the binomial expansion for $A = 1$, $B = x$, and $p = -1$. What is the series? Is it convergent?

Solution. The expression for this case is

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} C(-1,k)(-x)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)(-2)(-3)\cdots(-k)}{k!}(-x)^k \\
&= \sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\cdots\ . \qquad \text{(B-20)}
\end{aligned}
$$

This is the geometric series. The series converges for x in the range $-1 < x < 1$.